Gate Circuits with Feedback in Finite Algebras of Transients

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Abstract

Simulation of gate circuits is an efficient method of detecting hazards and oscillations that may occur in the presence of gate and wire delays. Ternary simulation, introduced by Eichelberger in 1965, consists of two algorithms, called A and B, and its results are well understood. Ternary simulation has been generalized by Brzozowski and Ésik in 2003 to an infinite algebra $\mathcal{C}$ and finite algebras $\mathcal{C}_k$, $k \geq 2$, where $\mathcal{C}_2$ is ternary algebra. Simulation in $\mathcal{C}$ has been studied extensively for feedback-free circuits, for which Algorithm A always terminates and Algorithm B is unnecessary. We study the simulation of gate circuits with feedback in finite algebras $\mathcal{C}_k$, in which Algorithms A and B always terminate. The Boolean functions of the gates are restricted to a set $\mathcal{G}$, where $\mathcal{G} = \mathcal{H} \cup \tilde{\mathcal{H}}$, $\mathcal{H} = \{\text{OR}, \text{XOR}\}$, and $\tilde{\mathcal{H}}$ is the set of functions obtained by complementing any number of inputs and/or the output of functions from $\mathcal{H}$. For gate functions restricted to $\mathcal{G}$, we prove that Algorithm B in Algebra $\mathcal{C}_k$, for $k > 2$ provides no more information than Algorithm B in ternary algebra.

1 Introduction

We study the following analysis problem for gate circuits. Suppose a circuit is started in a stable state and some of its inputs change once, and are kept indefinitely at their final values. We call this a *transition* of the circuit. We wait an appropriately long time [5] and observe the gate outputs. Some gates may remain in their initial states, others may change one or more times, and still others may *oscillate*, that is, they may take on the values 0 and 1 infinitely often.

Hazards are undesirable pulses that may occur during a transition under certain distributions of gate and wire delays. A *static hazard* exists in a gate if the gate output is the same before and after the transition, but takes on the complementary value temporarily during the transition. A *dynamic hazard* exists if a gate is supposed to change from its initial value to the complementary value only once, but changes several times instead. In general, hazards are undesirable, because they may lead to computation errors. Usually oscillations are also undesirable.

It is possible to detect hazards and oscillations using *binary analysis*, that is, analysis based on Boolean algebra [5]. However, such methods are exponential in the number of gates and wires. Consequently, several algebras have been proposed over the years to find an efficient *multi-valued simulation* method to detect these phenomena. For a recent survey of these algebras see [3].

The first algebra with more than two values used for hazard detection was ternary. Ternary simulation was introduced by Eichelberger [6], and uses the "uncertain" value Φ in addition to the binary values 0 and 1. It has two parts, called Algorithms A and B, which always terminate for any circuit; the resulting states of the gates after the algorithms are denoted $\mathbf{y}^A$ and $\mathbf{y}^B$, respectively. Ternary simulation is well understood [5], as we now explain.

The circuit is started in a stable binary state and some inputs change. The changing inputs are set to Φ in Algorithm A. The circuit is then analyzed in ternary algebra to determine whether some gate outputs become Φ as well. If a gate retains its binary value in $\mathbf{y}^A$, then it does not change during the transition. If it becomes Φ, then it is possible for it to change, depending on the relative delays in the circuit.

In Algorithm B, the circuit starts in state $\mathbf{y}^A$, and the inputs that are Φ are set to their final binary values. The circuit is again analyzed in ternary algebra. If a gate is Φ in $\mathbf{y}^B$, then it may have a nontransient oscillation – one that can persist indefinitely. If a gate has the same value in the initial state and in $\mathbf{y}^B$, but is Φ in $\mathbf{y}^A$, then it has a static hazard. Ternary simulation is unable to detect dynamic hazards.

In 2003, ternary simulation was generalized by Brzozowski and Ésik [2] to an infinite algebra $\mathcal{C}$ and finite algebras $\mathcal{C}_k$, for $k = 2, 3, \ldots$. These algebras, with and without small modifications, include all the successful multi-valued algebras used in the past, and $\mathcal{C}_2$ is ternary algebra. Algebra $\mathcal{C}$ uses the set of *transients* as the underlying set, where a transient is a nonempty word of alternating 0s and 1s.

Simulation in Algebra $\mathcal{C}$ was studied for feedback-free circuits [2, 4, 7, 8]. Algorithm A in $\mathcal{C}$ always terminates for these circuits, detects all hazards, permits us to count the number of signal changes occurring under worst-case conditions, and is polynomial in the number of gates. The results of the simulation are easily understood. For example, if a gate has the transient 1010 in $\mathbf{y}^A$, then the gate's initial value is 1, its final value is 0, and it may change three times during the transition.

For combinational circuits, simulation in Algebra $\mathcal{C}$ has been compared with binary analysis. It was shown in [4] that transient simulation covers binary analysis, meaning that it predicts all the changes that are possible in binary analysis. It is easy, however, to find examples of circuits in which transient simulation predicts more changes than binary analysis. The first step towards proving a converse of this result was made in [7], where it was shown that binary analysis covers simulation for feedback-free circuits of 1- and 2-input gates, if wire delays are taken into account. A more general result was proved in [8], where it was shown that binary analysis covers simulation for feedback-free circuits constructed with gates from $\mathcal{G}$, where $\mathcal{G} = \mathcal{H} \cup \tilde{\mathcal{H}}$, $\mathcal{H} = \{\text{OR}, \text{XOR}\}$, $\tilde{\mathcal{H}}$ is the set of functions obtained by complementing any number of inputs and/or the output of functions from $\mathcal{H}$, and OR and XOR can have arbitrary numbers of inputs. Note that $\mathcal{G}$ includes all the Boolean functions of two variables, except the two constant functions, as well as the commonly used AND, NOR, NAND and XNOR functions with any numbers of inputs. It was also shown in [8] that there are Boolean functions for which this result does not hold, if we add a constant number of delays per wire.

For a circuit with feedback, Algorithm A may not terminate, and Algorithm B is then not applicable. However, it is possible to use an algebra $\mathcal{C}_k$, for any $k > 1$, in which Algorithm A always terminates. The underlying set of values for $\mathcal{C}_k$ is the set of all transients of length less that k, plus Φ_k, a value that represents all the transients of length $\geq k$. Thus $\mathcal{C}_k$ approximates $\mathcal{C}$ in the sense that it detects all the transients up to length $k-1$ precisely, and lumps all longer transients into Φ_k.

Since Algorithm A always terminates in $\mathcal{C}_k$, Algorithm B is again applicable. In this paper we characterize the results of Algorithm B in Algebra $\mathcal{C}_k$, for $k > 2$. We prove that, for circuits with gates from $\mathcal{G}$, these results contain the same information as those obtained using the ternary algebra $\mathcal{C}_2$.

2 Gate Networks

Gate networks are defined in terms of directed graphs and Boolean functions.

We base our terminology and notation about graphs on that of [1]. A *digraph (directed graph)* D is an ordered triple $D = (V, E, \psi)$, where V is a nonempty set of *vertices*, E is a set (disjoint from V) of *arcs*, and ψ is an *incidence function* assigning to each arc of D an ordered pair of (not necessarily distinct) vertices of D. If $e \in E$, $u, v \in V$, and $\psi(e) = (u, v)$, then e *joins* u to v, $t(e) = u$ is the *tail* of e, and $h(e) = v$ is the *head*.

For $r \geq 1$, we define $[r] = \{1, \ldots, r\}$. A *directed walk* in D is a finite, nonempty sequence $W = v_0, e_1, v_1, \ldots, e_r, v_r$, whose terms are alternately vertices and arcs, such that for each $i \in [r]$, e_i joins v_{i-1} to v_i; v_0 is the *origin* of W, v_r is its *terminus*, and $v_1, \ldots, v_{r-1}$ are *internal vertices*. The *length* of a walk W is r; note that v_0 is a walk of length 0. A *directed trail* is a directed walk in which all the arcs are distinct. A *directed path* is a directed trail in which all the vertices are distinct. A directed walk is *closed* if $r \geq 1$ and $v_r = v_0$. A *directed cycle* is a closed directed trail whose origin and internal vertices are distinct.

Since we are dealing only with directed graphs, we simply use the terms *walk, trail, path, cycle* for *directed walk, etc.*

Suppose $D = (V, E, \psi)$ is a digraph. We assume that D has at least one vertex of indegree 0 and at least one vertex of indegree > 0. The vertices with indegree 0 are *(external) inputs*, and are labeled $x_1, \ldots, x_m$; let $V_x = \{x_1, \ldots, x_m\}$. The remaining vertices are labeled $y_1, \ldots, y_n$ and are *gates*. Let $V_y = \{y_1, \ldots, y_n\}$; then $V = V_x \cup V_y$. Thus $D = (V_x \cup V_y, E, \psi)$, where $V_x \cap V_y = \emptyset$.

Let $B = \{0, 1\}$ be the set of the two binary values. A *total state* is an assignment $s : V \to B$. A particular total state is denoted by an $(m+n)$-tuple $s = (s_1, \ldots, s_{m+n})$. The state s_i of vertex v_i in total state s is also denoted by $s(v_i)$. The *state of an arc* e from u to v in total state s is denoted by $s(e)$ and is defined as the state of vertex u, that is, $s(e) = s(t(e))$.

The arcs $e_{i_1}, \ldots, e_{i_{n_i}}$ such that $h(e_{i_1}) = \ldots = h(e_{i_{n_i}}) = y_i \in V_y$ are the n_i *inputs* of gate y_i, $n_i > 0$; these are all the arcs with head y_i. We assign a Boolean function $f_i : B^{n_i} \to B$ to each gate y_i; this is its *excitation function*.

A *(gate) network* N is a 4-tuple $N = (V_x \cup V_y, E, \psi, f)$, where $(V_x \cup V_y, E, \psi)$ is a digraph as above, and $f = (f_1, \ldots, f_n)$ is the n-tuple of excitation functions.

We denote OR by $\vee$, AND by $\wedge$, and complement by $^-$.

3 Extensions of Boolean Functions

A *transient* [2] is a nonempty binary word of alternating 0s and 1s, that is, it is an element of the set $\mathbf{T} = (01)^* \cup (10)^* \cup (01)^*0 \cup (10)^*1$. Transients are denoted by boldface letters. If $\mathbf{x}$ is a transient, then $\alpha(\mathbf{x})$, and $\omega(\mathbf{x})$ are the first and last letters of $\mathbf{x}$, respectively. Also $z(\mathbf{x})$ and $u(\mathbf{x})$ are the numbers of 0s and 1s in $\mathbf{x}$, respectively, and $l(\mathbf{x})$ is the length of $\mathbf{x}$.

We extend a Boolean function $f : B^n \to B$ to a Boolean function $\mathbf{f} : \mathbf{T}^n \to \mathbf{T}$ as follows. For an n-tuple $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ of transients, the digraph $F(\mathbf{x})$ of a Boolean function f has as vertices all the n-tuples $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of transients, where $\mathbf{v}_i$ is a nonempty prefix of $\mathbf{x}_i$, for all $i \in [n]$. There is an arc from vertex $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ to vertex $\mathbf{v}' = (\mathbf{v}'_1, \ldots, \mathbf{v}'_n)$ if and only if $\mathbf{v}$ and $\mathbf{v}'$ differ in exactly one coordinate, say i, and $\mathbf{v}'_i = \mathbf{v}_i c$, where $c \in B$. Graph $F(\mathbf{x})$ shows all possible orders in which the n variables can change along paths from the initial vertex $(\alpha(\mathbf{x}_1), \ldots, \alpha(\mathbf{x}_n))$ to the final vertex $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$. Moreover, with each vertex $\mathbf{v}$ we associate an output $f(\omega(\mathbf{v}_1), \ldots, \omega(\mathbf{v}_n))$.

A *contraction* of a binary word w is a transient obtained by replacing all sequences of consecutive 0s by a single 0 and all sequences of consecutive 1s by a single 1. For example, the contraction of 011000111 is 0101. In the graph $F(\mathbf{x})$ for a Boolean function f, the *output of a path* π is the sequence of outputs of the vertices of π; it is denoted as $\mathbf{w}(\pi)$. The *transient of a path* π is the contraction $\mathbf{z}(\pi)$ of $\mathbf{w}(\pi)$. The value of the extension $\mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ of f is the longest possible path transient in $F(\mathbf{x})$. It represents the longest transient that might occur during the input change from $(\alpha(\mathbf{x}_1), \ldots, \alpha(\mathbf{x}_n))$ to $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$.

3.1 The Role of Complements

Proposition 1 *Suppose $n \geq 1$ and $f, g : B^n \to B$ are Boolean functions. If, for all $x_1, \ldots, x_n \in B^n$, $f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = g(x_1, \ldots, x_{i-1}, \overline{x_i}, x_{i+1}, \ldots, x_n)$, then, for all $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbf{T}$, $\mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n) = \mathbf{g}(\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \overline{\mathbf{x}_i}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n)$.*

Proof: Let $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n)$ and $\mathbf{x}' = (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \overline{\mathbf{x}_i}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n)$. For each path π in $F(\mathbf{x})$ there is a path π' in $G(\mathbf{x}')$, obtained from π by complementing the ith component in each vertex of π, such that $\mathbf{w}(\pi) = \mathbf{w}(\pi')$, and vice versa. Thus $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{x}')$. $\square$

This proposition permits us to consider a gate realizing the function $\mathbf{f}$ as a composition of an inverter for $\mathbf{x}_i$ and a gate realizing $\mathbf{g}$.

Proposition 2 *Let $n \geq 1$ and let $f, g : B^n \to B$ be Boolean functions. If, for all $x_1, \ldots, x_n \in B^n$, $f(x_1, \ldots, x_n) = \overline{g}(x_1, \ldots, x_n)$, then for all $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbf{T}$, $\mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \overline{\mathbf{g}}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$.*

Proof: For each path π in $F(\mathbf{x})$, the identical path π' in $G(\mathbf{x})$ has the property that $\mathbf{w}(\pi) = \overline{\mathbf{w}}(\pi')$, and vice versa. Thus $\mathbf{f}(\mathbf{x}) = \overline{\mathbf{g}}(\mathbf{x})$. $\square$

This proposition permits us to consider a gate realizing the function $\mathbf{f}$ as a composition of a gate realizing $\mathbf{g}$ followed by an inverter.

3.2 Dominant Input Values

Definition 1 *Let $n > 1$ and let $f : B^n \to B$ be a Boolean function. A value $d \in B$ of argument x_i of f is* dominant for f *if the value of f is independent of the other arguments when $x_i = d$, that is, if $f(x_1, \dots, x_{i-1}, d, x_{i+1}, \dots, x_n) = f(x'_1, \dots, x'_{i-1}, d, x'_{i+1}, \dots, x'_n)$, for all $x_j, x'_j \in B^{n-1}$, $j \in [n], j \neq i$. We denote the value of f when $x_i = d$ by $f_{x_i=d}$. If $\mathbf{f}$ is the extension of f, then a transient value $\mathbf{t}$ of argument $\mathbf{x}_i$ is* dominant for $\mathbf{f}$ *if the value of $\mathbf{f}$ is independent of the other arguments when $\mathbf{x}_i = \mathbf{t}$.*

For example, if $f(x_1, x_2) = \overline{x_1} \vee x_2$, then $x_1 = 0$ and $x_2 = 1$ are dominant.

Proposition 3 *Let $n \geq 1$, let $f : B^n \to B$ be a Boolean function that depends on each of its arguments. If none of the $\mathbf{x}_i$ is a single letter, then the length of $\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is at least the maximum of the lengths of the $\mathbf{x}_i$.*

Proposition 3 was proved in [2]; we now use it to prove the following result:

Proposition 4 *Let $n > 1$, let $f : B^n \to B$ be a Boolean function that depends on each of its arguments. A transient value $\mathbf{x}_i = \mathbf{t}$ is dominant for $\mathbf{f}$ if and only if it is binary, say $\mathbf{t} = d$, and $x_i = d$ is dominant for f. Also, if $\mathbf{x}_i = d$ is dominant for $\mathbf{f}$, then $\mathbf{f}_{\mathbf{x}_i=d} = f_{x_i=d}$.*

Proof: Suppose $\mathbf{t} \notin B$ is dominant for $\mathbf{f}$; then

$$\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{t}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n) = \mathbf{f}(\mathbf{x}'_1, \dots, \mathbf{x}'_{i-1}, \mathbf{t}, \mathbf{x}'_{i+1}, \dots, \mathbf{x}'_n) = \mathbf{s},$$

for all $(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n), (\mathbf{x}'_1, \dots, \mathbf{x}'_{i-1}, \mathbf{x}'_{i+1}, \dots, \mathbf{x}'_n) \in \mathbf{T}^{n-1}$, and some $\mathbf{s} \in \mathbf{T}$. In particular, this can occur when none of the arguments is a single letter and one argument, $\mathbf{x}_j$, $j \neq i$, has length greater than $l(\mathbf{s})$. By Proposition 3, the length of $\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ must be greater than $l(\mathbf{s})$, which is a contradiction. Hence $\mathbf{t}$ must be binary: say $\mathbf{t} = b$. Since $\mathbf{f}$ agrees with f on binary values, it is clear that $x_i = b$ is dominant for f, and that $\mathbf{f}_{\mathbf{x}_i=b} = f_{x_i=b}$ is binary.

Conversely, suppose that $x_i = b$ is dominant for f. In every vertex $(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, b, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n)$ of the digraph $F(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, b, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n)$ the ith component is b. Hence the value

$$f(\omega(\mathbf{v}_1), \dots, \omega(\mathbf{v}_{i-1}), \omega(b), \omega(\mathbf{v}_{i+1}), \dots, \omega(\mathbf{v}_n))$$

of the output associated with every vertex is $f_{x_i=b}$. Thus every path in this graph has the transient $f_{x_i=b}$ of length 1, and hence $\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, b, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n) = f_{x_i=b}$. Therefore $\mathbf{x}_i = b$ is dominant for $\mathbf{f}$. $\square$

Remark 1 *Definition 1 and Proposition 4 can be extended to* tuples *of argument values. For example, consider the function $f(x_1, x_2, x_3) = (x_1 \wedge \overline{x_3}) \vee (x_2 \wedge x_3)$, which is not in $\mathcal{G}$. If $x_1 = 1$ and $x_2 = 1$; then $f(1, 1, x_3) = 1$, independently of x_3, but neither $x_1 = 1$ nor $x_2 = 1$ is dominant.*

In general, for $h < n$, we define an h-tuple t of argument values of a Boolean function of n variables to be dominant *if the value of the function depends only on the values in the h-tuple, and no subtuple of t has this property.*

Note that the OR *function with two or more arguments has only dominant 1-tuples, namely, every $x_i = 1$ is a dominant value. The* XOR *function with two or more arguments has no dominant tuples. It follows that every function in $\mathcal{G}$ with two or more arguments has either no dominant tuples or only dominant 1-tuples.* $\square$

In what follows, for reasons that become clearer later, we single out one of the inputs, say $\mathbf{x}_i$, of a function $f : \mathbf{T}^n \to \mathbf{T}$, $n > 1$, as the *main input,* and the remaining inputs are called *side inputs*. We study the relation between the length of $\mathbf{x}_i$ and the length of $\mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ with the side inputs as parameters.

3.3 Extension of the OR Function

An OR function of n variables is 1 if at least one of the variables is 1; thus a 1-argument OR function is the identity function. Recall [2] that, if $f : B^n \to B$ is the OR function, $n \geq 1$, then $\mathbf{y} = \mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is the word in $\mathbf{T}$ determined by the conditions

$$\begin{aligned}
\alpha(\mathbf{y}) &= \alpha(\mathbf{x}_1) \vee \ldots \vee \alpha(\mathbf{x}_n) && (1)\\
\omega(\mathbf{y}) &= \omega(\mathbf{x}_1) \vee \ldots \vee \omega(\mathbf{x}_n) && (2)\\
z(\mathbf{y}) &= \begin{cases} 0 & \text{if } \exists h \in [n]\ \mathbf{x}_h = 1 \\ 1 + \sum_{h=1}^{n}(z(\mathbf{x}_h) - 1) & \text{otherwise.} \end{cases} && (3)
\end{aligned}$$

Lemma 1 *If $f : B^n \to B$, $n > 1$, is the* OR *function, then*

1. *$l(\mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n)) < l(\mathbf{x}_i)$, if and only if $l(\mathbf{x}_i) > 1$ and one of the side inputs is* 1.
2. *$l(\mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n)) = l(\mathbf{x}_i)$, if and only if one of the following conditions holds:*
   - *(a)* $\mathbf{x}_i = 1$.
   - *(b)* $\mathbf{x}_i = 0$ *and one (or more) of the side inputs is 1.*
   - *(c)* $\mathbf{x}_i = 0$ *and all the side inputs are 0.*

   (d) $\mathbf{x}_i = (01)^i$, *$i > 0$, and all the side inputs are either* 01 *or* 0.
   (e) $\mathbf{x}_i = (01)^i0$, *$i > 0$, and all the side inputs are* 0.
   (f) $\mathbf{x}_i = (10)^i$, *$i > 0$, and all the side inputs are either* 10 *or* 0.
   (g) $\mathbf{x}_i = (10)^i1$, *$i > 0$, and all the side inputs are* 101, 01, 10 *or* 0.

3. $l(\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n)) > l(\mathbf{x}_i)$ *if and only if the conditions in Parts 1 and 2 are not satisfied.*

Proof: See Appendix.

3.4 Extension of the XOR Function

A XOR function of n variables is 1 if an odd number of the variables is 1; thus a 1-argument XOR function is the identity function. Recall [2] that, if $n \geq 1$ and $f : B^n \to B$ is the XOR function, then $\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is the word $\mathbf{y}$ in $\mathbf{T}$ satisfying the conditions

$$\alpha(\mathbf{y}) = f(\alpha(\mathbf{x}_1), \dots, \alpha(\mathbf{x}_n)) \tag{4}$$
$$\omega(\mathbf{y}) = f(\omega(\mathbf{x}_1), \dots, \omega(\mathbf{x}_n)) \tag{5}$$
$$l(\mathbf{y}) = 1 + \sum_{h=1}^{n}(l(\mathbf{x}_h) - 1). \tag{6}$$

Lemma 2 *If $f : B^n \to B$, $n > 1$, is the* XOR *function, then*

1. $l(\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n)) = l(\mathbf{x}_i)$, *if and only if all the side inputs are binary.*
2. $l(\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n)) > l(\mathbf{x}_i)$ *if and only if at least one side input is not binary.*

Proof: This follows directly from Equation (6). □

3.5 Extensions of Functions in $\mathcal{G}$

By duality with the OR function, if $f : B^n \to B$ is the AND function, then $\mathbf{y} = \mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{T}$ is given by

$$\alpha(\mathbf{y}) = \alpha(\mathbf{x}_1) \wedge \dots \wedge \alpha(\mathbf{x}_n) \tag{7}$$
$$\omega(\mathbf{y}) = \omega(\mathbf{x}_1) \wedge \dots \wedge \omega(\mathbf{x}_n) \tag{8}$$
$$u(\mathbf{y}) = \begin{cases} 0 & \text{if } \exists i \in [n]\ \mathbf{x}_i = 0 \\ 1 + \sum_{i=1}^{n}(u(\mathbf{x}_i) - 1) & \text{otherwise.} \end{cases} \tag{9}$$

The extension of the NOR (NAND, XNOR) function of any number of arguments is the complement of the extension of the OR (AND, XOR) function. Note that function composition does not preserve extensions in general [2].

The following results are consequences of Lemmas 1 and 2, and Propositions 1 and 2:

Proposition 5 *If $n > 1$ and $f : B^n \to B$ is in $\mathcal{G}$, then $l(\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n)) < l(\mathbf{x}_i)$ if and only if $l(\mathbf{x}_i) > 1$ and one of the side-input values of $\mathbf{f}$ is dominant.*

Corollary 1 *Let $n > 1$, and let $f : B^n \to B$ be in $\mathcal{G}$. If $\mathbf{f}$ has no dominant side-input values, and $l(\mathbf{x}_i) \geq k > 1$, then $l(\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n)) \geq k$.*

4 Simulation in Algebra $\mathcal{C}_k$

The *change-counting algebra* [2] (also called *algebra of transients* [4, 7]) is defined as

$$\mathcal{C} = (\mathbf{T}, \vee, \wedge, ^{-}, 0, 1),$$

where $\vee$ is defined by Equations 1–3, $\wedge$, by Equations 7–9, and the complement of a transient $\mathbf{t} = c_1 \dots c_r$, where $c_i \in B$, is $\overline{\mathbf{t}} = \overline{c_1} \dots \overline{c_r}$.

We now study the simulation of circuits constructed with gates from $\mathcal{G}$ in Algebra $\mathcal{C}_k$. Recall [2] that, for $k \geq 2$, relation $\sim_k$ is defined on the set $\mathbf{T}$ of transients as follows: For $\mathbf{x}, \mathbf{x}' \in \mathbf{T}$, $\mathbf{x} \sim_k \mathbf{x}'$ if either $\mathbf{x} = \mathbf{x}'$ or $\mathbf{x}$ and $\mathbf{x}'$ are both of length $\geq k$. This relation is a congruence on $\mathcal{C}$, and the quotient algebra $\mathcal{C}_k$ is defined over the set $\mathbf{T}_k = \{[\mathbf{x}] \mid l(\mathbf{x}) < k\} \cup \{\Phi_k\}$, where $\Phi_k = \{x \mid l(x) \geq k\}$. Since all the classes except Φ_k are singletons, we simply write $\mathbf{T}_k = \{\mathbf{x} \mid l(\mathbf{x}) < k\} \cup \{\Phi_k\}$, and refer to elements of $\mathbf{T}_k$ as transients.

Every function $\mathbf{f} : \mathbf{T}^n \to \mathbf{T}$ in $\mathcal{C}$ is now also defined in $\mathcal{C}_k$ as follows:

$$\mathbf{f}([\mathbf{x}_1], \dots, [\mathbf{x}_n]) = [\mathbf{f}(\mathbf{x}'_1, \dots, \mathbf{x}'_n)],$$

where $\mathbf{x}'_i$ is any representative of the class $[\mathbf{x}_i]$, for $i \in [n]$.

We also define the *prefix order* $\leq$ on $\mathbf{T}_k$: $[\mathbf{x}'] \leq [\mathbf{x}]$ if either (a) $[\mathbf{x}']$ and $[\mathbf{x}]$ are singleton classes and $\mathbf{x}'$ is a prefix of $\mathbf{x}$, or (b) $[\mathbf{x}] = \Phi_k$. The length $l([\mathbf{x}])$ of a class $[\mathbf{x}]$ is $l(\mathbf{x})$ if $[\mathbf{x}] = \{\mathbf{x}\}$, and $l(\Phi_k) = k$.

The *suffix order* $\preceq$ is defined similarly.

We define an operation $\circ$ as follows. For $c, d \in B$, if $c = d$, then $c \circ d = c$. For $c \neq d$, if the simulation is done in algebra $\mathcal{C}_2$, which is ternary algebra, then $c \circ d = d \circ c = \Phi_2$. Otherwise, if $c \neq c$, and the simulation uses algebra $\mathcal{C}$ or algebra $\mathcal{C}_k$ with $k > 2$, then $c \circ d = cd$, where cd represents the concatenation of c and d. This notation is extended to tuples. If $\hat{a} = (\hat{a}_1, \dots, \hat{a}_m)$ and $a = (a_1, \dots, a_m)$

$$\hat{a} \circ a = (\hat{a}_1 \circ a_1, \dots, \hat{a}_m \circ a_m).$$

For example, $(1, 0, 0, 1) \circ (1, 1, 0, 0) = (1, \Phi_2, 0, \Phi_2)$ in $\mathcal{C}_2$; otherwise, $(1, 0, 0, 1) \circ (1, 1, 0, 0) = (1, 01, 0, 10)$.

A network is initially in the stable total state $(\hat{a}, b)$, that is, $f(\hat{a}, b) = b$, and then the input m-tuple $\hat{a}$ changes to a.

Algorithm A is shown below, where $\mathbf{f} : \mathbf{T}_k^{m+n} \to \mathbf{T}_k^n$ is the tuple of excitation functions of the network. Each gate variable is non-decreasing in the prefix order in this algorithm, and the algorithm always terminates. Its result is denoted by $\mathbf{y}^A$. It follows from [2] that $\mathbf{y}^A$ is the least fixed point of the function $\mathbf{f}(\mathbf{a}, \mathbf{x})$ over b with respect to the prefix order, i.e.,

$$\mathbf{f}(\mathbf{a}, \mathbf{y}^A) = \mathbf{y}^A, \ \textit{and} \tag{10}$$

$$\mathbf{y} \geq b \ \& \ \mathbf{f}(\mathbf{a}, \mathbf{y}) = \mathbf{y} \ \ \Rightarrow \ \ \mathbf{y}^A \leq \mathbf{y}. \tag{11}$$

Algorithm A

```
$h := 0$;
$\mathbf{a} := \hat{a} \circ a$;
$\mathbf{y}^0 := b$;
**repeat**
    $h := h + 1$;
    $\mathbf{y}^h := \mathbf{f}(\mathbf{a}, \mathbf{y}^{h-1})$;
**until** $\mathbf{y}^h = \mathbf{y}^{h-1}$;
```

In Algorithm B below, each gate variable is non-increasing in the suffix order. The result of the algorithm is denoted by $\mathbf{y}^B$. It follows from [2] that Algorithm B in Algebra $\mathcal{C}_k$ computes the greatest fixed point of function $\mathbf{f}(a, \mathbf{x})$ below $\mathbf{y}^A$ with respect to the suffix order, that is

$$\mathbf{f}(a, \mathbf{y}^B) = \mathbf{y}^B, \ \textit{and} \tag{12}$$

$$\mathbf{y} \preceq \mathbf{y}^A \ \& \ \mathbf{f}(a, \mathbf{y}) = \mathbf{y} \ \ \Rightarrow \ \ \mathbf{y}^B \succeq \mathbf{y}. \tag{13}$$

Algorithm B

```
$h := 0$;
$\mathbf{y}^0 := \mathbf{y}^A$;
**repeat**
    $h := h + 1$;
    $\mathbf{y}^h := \mathbf{f}(a, \mathbf{y}^{h-1})$;
**until** $\mathbf{y}^h = \mathbf{y}^{h-1}$;
```

5 Active Paths and Cycles

In this section the total state of a network is an $(m+n)$-tuple $\mathbf{s} = (\mathbf{s}_1, \dots, \mathbf{s}_{m+n})$ of transients. The state of vertex v_i in $\mathbf{s}$ is $\mathbf{s}(v_i) = \mathbf{s}_i$, and the state of arc e from u to v is $\mathbf{s}(e) = \mathbf{s}(t(e))$.

In the sequel, we consider three stable total states: the initial state $(\hat{a}, b)$, the state $(\mathbf{a}, \mathbf{y}^A)$ at the end of Algorithm A, and the state $(a, \mathbf{y}^B)$ at the end of Algorithm B. To simplify the terminology we refer to these as stable states b, $\mathbf{y}^A$ and $\mathbf{y}^B$.

Let $T = v_0, e_1, y_1, \dots, e_r, y_r$ be a path or cycle in a gate network, where v_0 is an input or a gate, and $y_1, \dots, y_r$ are gates. Any arc joining some node $w \notin \{v_0, y_1, \dots, y_r\}$ to a node y_i, $1 \le i \le r$ is called an *side input* of gate y_i. Let $e'_i = e'_{i_1}, \dots, e'_{i_{n_i-1}}$ be the $(n_i - 1)$-tuple of side inputs of gate y_i; thus the inputs of gate y_i are the side inputs together with the *main input* e_i, which is the arc from v_0 to y_1, if $i = 1$, and the arc from y_{i-1} to y_i, otherwise. There is no loss of generality in assuming that the main input is the last input of y_i.

Definition 2 *An arc e from u to v is* active *in stable state* $\mathbf{s}$ *if no side input to gate v is dominant in* $\mathbf{s}$*; otherwise, e is* inactive*. A path or cycle $T = v_0, e_1, v_1, \dots, e_r, v_r$ is* active *in stable state* $\mathbf{s}$ *if all of its arcs are active. We say that vertex v_i* activates *vertex v_j in stable state* $\mathbf{s}$ *if there is an active path or cycle of length greater than 0 from v_i to v_j.*

Clearly, 'activates' is a transitive relation. Note that an arc leading to an identity gate (1-input OR gate) or an inverter is always active, since such a gate has no side inputs. Also, an arc leading to a XOR gate is always active, since that gate has no dominant inputs.

Consider a cycle $C = v_0, e_1, v_1, \dots, e_r, v_0$; if the cycle is active in some state, then every vertex activates itself, and 'activates' is reflexive. Also, if v_i and v_j are two vertices in a cycle, then v_i activates v_j, and *vice versa*; thus 'activates' is also symmetric, and hence an equivalence relation.

Proposition 6 *Let N be a network with excitation functions in $\mathcal{G}$. If N is in a stable state* $\mathbf{s}$ *and C is an active cycle, then all the vertices in C have transients of the same length.*

Proof: If the length of a transient in the cycle increases at some vertex, then it must also decrease at another vertex, since the cycle is finite. Proposition 5 shows that the length cannot decrease by going from vertex v_{i-1} to vertex v_i if v_i is a gate without dominant side inputs. Hence the lengths must be equal. □

6 Interior Values

Let $k \geq 2$; a value $\mathbf{x} \in \mathbf{T}_k$ is *exterior* if $\mathbf{x} \in \{0, 1, \Phi_k\}$; otherwise, it is *interior*. In the case of ternary algebra, $k = 2$ and there are no interior values.

Lemma 3 *If a vertex has an interior value in $\mathbf{y}^B$, then it has the same value in $\mathbf{y}^A$.*

Proof: Since Algorithm B is non-increasing in the suffix order, if a variable has a value $\mathbf{s}_i$ in $\mathbf{y}^A$ and a value $\mathbf{t}_i$ in $\mathbf{y}^B$, then $\mathbf{t}_i$ is a nonempty suffix of $\mathbf{s}_i$. If $\mathbf{y}^B(v_i)$ is interior, and $\mathbf{y}^A(v_i) \neq \mathbf{y}^B(v_i)$, then the length of v_i must decrease during Algorithm B. Thus suppose the value of v_i is the same at step $j-1$ of Algorithm B as it is in $\mathbf{y}^A$, but the length of v_i decreases in step j. By Proposition 5, v_i has a dominant input in step $j-1$. But then v_i also has that dominant input in $\mathbf{y}^B$, since a binary value cannot change in Algorithm B. This would make $\mathbf{y}^B(v_i)$ binary, contradicting that $\mathbf{y}^B(v_i)$ is interior. Thus the value of v_i in $\mathbf{y}^A$ must be the same as in $\mathbf{y}^B$. □

Let I be the set of all vertices of a network N that have interior values in $\mathbf{y}^B$. For $u, v \in I$, let $u \sim v$ if $u = v$ or u activates v and v activates u. Clearly, $\sim$ is an equivalence relation on I. Equivalence class $[u]$ *activates* equivalence $[v]$, if and only if $u = v$ or u activates v. The relation 'activates' on $I/\sim$ is a partial order. An equivalence class $[u]$ is *primary* in this partial order if $[v]$ activates $[u]$ implies $[v] = [u]$. Since we are dealing with finite network graphs, there always exists at least one primary class. A primary class cannot consist of a single state that does not appear in any cycle, because such a vertex must depend only on the external network inputs, and its value is binary in $\mathbf{y}^B$.

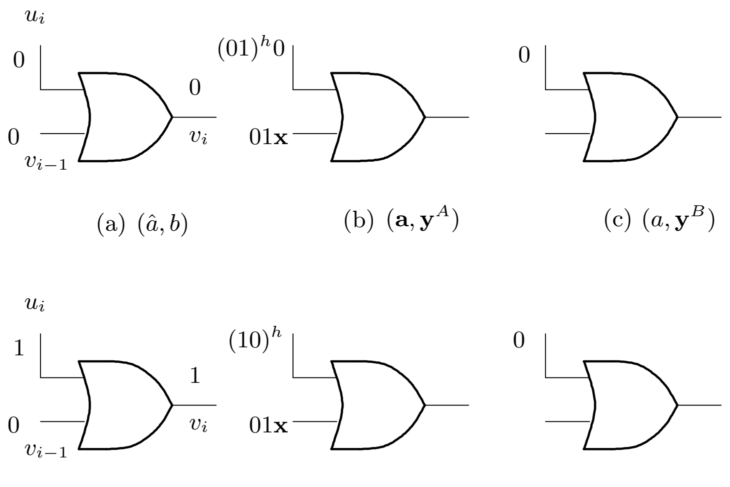
Theorem 1 *For every $k > 2$, the result of Algorithm B is the same in $\mathcal{C}_k$ as in ternary simulation.*

Proof: Let P be a primary equivalence class of interior values in $\mathbf{y}^B$. All the vertices of P are binary in state b, but interior in $\mathbf{y}^B$ and, by Lemma 3, also interior in $\mathbf{y}^A$. Suppose the values of all the vertices in P are the same at step $j-1$ of Algorithm A as they are in b, but v_i is interior in step j. Then at least one of the vertices, say $u_i \notin P$, joined by an arc to v_i must have an interior value in step $j-1$, and also in $\mathbf{y}^A$. (This value can't be Φ_k, because then v_i would be Φ_k in $\mathbf{y}^A$.)

Now $\mathbf{y}^B(u_i)$ cannot be Φ_k, for this would force the values in P to be Φ_k, by Corollary 1. It cannot be interior, for then it would be in another equivalence class Q that activates P, contradicting that P is primary. Thus $\mathbf{y}^B(u_i)$ must be binary. Since Algorithm B is non-increasing in the suffix order, $\mathbf{y}^A(u_i)$ must end in $\mathbf{y}^B(u_i)$. Since Algorithm A is non-decreasing in the prefix order, $\mathbf{y}^A(u_i)$ must begin with $b(u_i)$.

Suppose first that v_i is an OR gate. Because v_i belongs to an active cycle in $\mathbf{y}^B$, we have $\mathbf{y}^B(u_i) = 0$, as shown in Fig. 1(c) and (f). (The gate may have other inputs that are not shown; all such inputs must be 0 in $\mathbf{y}^B$.) Let the in-cycle predecessor of v_i be v_{i-1}; in case the cycle consists of a single state, we have $v_i = v_{i-1}$. Clearly, v_{i-1} is also in P, and cannot be 1 in state b, because this would prevent v_i from changing. Therefore $b(v_{i-1}) = 0$, as shown in Fig. 1(a) and (d).

Case 1: $b(v_i) = 0$. Since $(\hat{a}, b)$ is stable, $b(u_i) = 0$, as in Fig. 1(a). Since $\mathbf{y}^A(u_i)$ must begin with $b(u_i)$, end with $\mathbf{y}^B(u_i)$ and be of length at least 2, $\mathbf{y}^A(u_i)$ has the form $(01)^h0$, where $h \geq 1$, as in Fig. 1(b). Also, $\mathbf{y}^A(v_{i-1})$ must begin with $b(v_{i-1})$, and be of length at least 2; thus v_{i-1} begins with 01. Since v_i has no dominant inputs, Part 1 of Lemma 1 does not apply. Also, the inputs $01\mathbf{x}$ and $(01)^h0$ do not fit any of the patterns of Part 2. Thus Part 3 applies, v_i cannot be stable in $\mathbf{y}^A$, and this case cannot occur.



(a) $(\hat{a}, b)$ (b) $(\mathbf{a}, \mathbf{y}^A)$ (c) $(a, \mathbf{y}^B)$

(d) $(\hat{a}, b)$ (e) $(\mathbf{a}, \mathbf{y}^A)$ (f) $(a, \mathbf{y}^B)$

Figure 1: OR gate in an interior cycle.

Case 2: $b(v_i) = 1$. Then there is a vertex, say u_i, connected to v_i by a side-input arc, that is initially 1 (there may be other such vertices) and is interior in step $j-1$ (as must be the other such vertices). Now $\mathbf{y}^A(u_i)$ begins with 1, ends with 0 and is of length at least 2; thus it has the form $(10)^h$, where $h \geq 1$. As in Case 1, v_{i-1} must be of the form $01\mathbf{x}$. Part 1 of Lemma 1 does not apply. If $h > 1$, then Part 2 does not apply. Thus v_i cannot be stable in $\mathbf{y}^A$, and this case cannot occur.

There remains the possibility that $h = 1$, that is, $\mathbf{y}^A(u_i) = 10$. By Part 2 of Lemma 1, v_{i-1} must be of the form $(10)^h$ or $(10)^h1$, $h \geq 1$, for in all other cases v_i cannot be stable in $\mathbf{y}^A$. This contradicts that v_{i-1} must begin with 0. Thus this case cannot occur.

In summary, the gate which changes first in Algorithm A cannot be an OR gate.

Now suppose that v_i is a XOR gate; then it has no dominant inputs. Since $\mathbf{y}^A(u_i)$ is interior, by Lemma 2, the cycle cannot be stable in $\mathbf{y}^A$. This is a contradiction.

The argument for other functions in $\mathcal{G}$ follows from Propositions 1 and 2.

Altogether, we have shown that no variable can have an interior value in $\mathbf{y}^B$. It is now easy to verify that Algorithm B in $\mathcal{C}_2$ produces the same value for each variable as Algorithm B in $\mathcal{C}_k$ for every $k > 2$. □

7 An Example

To illustrate the simulation of circuits in Algebras $\mathcal{C}_k$ we give three simulations of the circuit of Figure 2. The circuit has the following excitation equations:

$$\mathbf{f}_1 = \overline{\mathbf{x}},\ \mathbf{f}_2 = \mathbf{x} \wedge \mathbf{y}_1,\ \mathbf{f}_3 = \mathbf{x} \vee \mathbf{y}_2,\ \mathbf{f}_4 = \mathbf{y}_3 \vee \mathbf{y}_4,\ \mathbf{f}_5 = \overline{\mathbf{y}_4},\ \mathbf{f}_6 = \mathbf{y}_4 \wedge \mathbf{y}_5,\ \mathbf{f}_7 = \mathbf{y}_6 \vee \mathbf{y}_7.$$

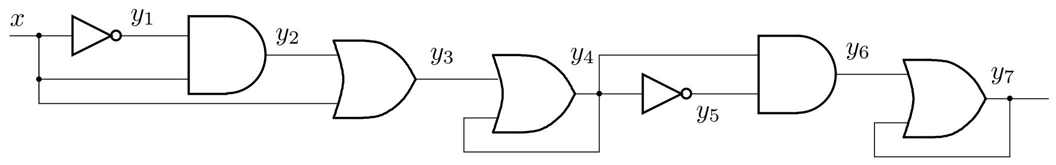


Figure 2: Circuit 1.

The simulations in $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_5$ are shown in Tables 1—3. From Algorithm A of ternary simulation we learn that each variable may change during the transition. From Algorithm B we know that $\mathbf{y}_1$ changes from 1 to 0 and $\mathbf{y}_3$ from 0 to 1, but we don't know whether these changes contain dynamic hazards. There is a static hazard in $\mathbf{y}_2$, and $\mathbf{y}_4$ to $\mathbf{y}_7$ may oscillate indefinitely.

Table 1: Simulation in $\mathcal{C}_2$.

	$\mathbf{x}$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{y}_7$
initial state	0	1	0	0	0	1	0	0
	Φ_2	1	0	0	0	1	0	0
	Φ_2	Φ_2	Φ_2	Φ_2	0	1	0	0
	Φ_2	Φ_2	Φ_2	Φ_2	Φ_2	1	0	0
	Φ_2	Φ_2	Φ_2	Φ_2	Φ_2	Φ_2	Φ_2	0
result A_2	Φ_2	Φ_2	Φ_2	Φ_2	Φ_2	Φ_2	Φ_2	Φ_2
	1	Φ_2	Φ_2	Φ_2	Φ_2	Φ_2	Φ_2	Φ_2
	1	0	Φ_2	1	Φ_2	Φ_2	Φ_2	Φ_2
result B_2	1	0	0	1	Φ_2	Φ_2	Φ_2	Φ_2

Table 2: Simulation in $\mathcal{C}_3$.

	$\mathbf{x}$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{y}_7$
initial state	0	1	0	0	0	1	0	0
	01	1	0	0	0	1	0	0
	01	10	01	01	0	1	0	0
	01	10	Φ_3	01	01	1	0	0
	01	10	Φ_3	Φ_3	01	10	01	0
	01	10	Φ_3	Φ_3	Φ_3	10	Φ_3	01
result A_3	01	10	Φ_3	Φ_3	Φ_3	Φ_3	Φ_3	Φ_3
	1	10	Φ_3	Φ_3	Φ_3	Φ_3	Φ_3	Φ_3
	1	0	10	1	Φ_3	Φ_3	Φ_3	Φ_3
result B_3	1	0	0	1	Φ_3	Φ_3	Φ_3	Φ_3

From the simulation in $\mathcal{C}_3$, we obtain the new knowledge that the change in $\mathbf{y}_1$ is free of hazards, whereas that in $\mathbf{y}_3$ has a dynamic hazard, but we don't know how many changes take place.

Algorithm A in $\mathcal{C}_5$ gives the additional information that $\mathbf{y}_2$ changes exactly twice, and $\mathbf{y}_5$, exactly three times. $\Box$

Table 3: Simulation in $\mathcal{C}_5$.

	$\mathbf{x}$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{y}_7$
initial state	0	1	0	0	0	1	0	0
	01	1	0	0	0	1	0	0
	01	10	01	01	0	1	0	0
	01	10	010	01	01	1	0	0
	01	10	010	0101	01	10	01	0
	01	10	010	0101	0101	10	010	01
	01	10	010	0101	Φ_5	1010	Φ_5	0101
result A_5	01	10	010	0101	Φ_5	Φ_5	Φ_5	Φ_5
	1	10	010	0101	Φ_5	Φ_5	Φ_5	Φ_5
	1	0	10	1	Φ_5	Φ_5	Φ_5	Φ_5
result B_5	1	0	0	1	Φ_5	Φ_5	Φ_5	Φ_5

8 Conclusions

It follows from Theorem 1 that only the following three types of simulation results are possible:

1. Algorithm A terminates in $\mathcal{C}$; Algorithm B is unnecessary. There may be static and dynamic hazards, but there are no oscillations possible. Every feedback-free circuit has this type of behavior. Its behavior is combinational, meaning that, if one waits long enough, all the variables will have binary values. Some variables may have a number of changes, but these are just hazards and they are of bounded length.

2. Algorithm A does not terminate in $\mathcal{C}$. For any k, after Algorithm A in $\mathcal{C}_k$, Algorithm B results in binary values only. This means the circuit has feedback, but the behavior is combinational, that is, the feedback is degenerate in this case. Hazards may, of course, exist; in fact, they may be of arbitrary length for the variables that are Φ_k in Algorithm B. To get a longer hazard, just increase the k.

3. Algorithm A does not terminate in $\mathcal{C}$. For any k, after Algorithm A in $\mathcal{C}_k$, Algorithm B results in at least one Φ_k. Thus, for some variables, the behavior may be as in Case 2 above, but at least one variable is involved in a nontransient oscillation [5].

In the case of Algorithm B in $\mathcal{C}_2$, a value of Φ means that there exists a nontransient cycle [5] in the binary analysis of the circuit if wire delays are taken into account. Our theorem shows that no further comparison of simulation in $\mathcal{C}_k$ to binary analysis is needed, since ternary simulations already covers all the cases.

9 Appendix

Proof of Lemma 1: All the input conditions are divided into three disjoint classes. If any given input conditions do not hold in one part, then they hold in one of the other two parts. Thus it is sufficient to prove only the "if" parts of the three claims.

1. If $l(\mathbf{x}_i) > 1$ and a side input is 1, then $\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n) = 1$, and the claim follows.
2. (a) If $\mathbf{x}_i = 1$, then $\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n) = 1$ and the lengths are equal.

   (b) If $\mathbf{x}_i = 0$ and a side input is 1, then $\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n) = 1$.

   (c) If all inputs are 0 then $\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n) = 0$.

   Using Equations (1)—(3), it is easy to verify that the following hold:

   (d) $(01)^i \vee 01 \vee 0 = (01)^i$.

   (e) $(01)^i 0 \vee 0 = (01)^i 0$.

   (f) $(10)^i \vee 10 \vee 0 = (10)^i$.

   (g) $(10)^i 1 \vee 101 \vee 01 \vee 10 \vee 0 = (10)^i 1$.

   In each of the four cases above, the side inputs contribute nothing to the number of 0s in the result, and to its initial and final letters; these are determined entirely by $\mathbf{x}_i$. Hence the lengths are equal in all these cases.
3. By 2(a), we can assume that $\mathbf{x}_i \neq 1$. By 2(b) and (c), if $\mathbf{x}_i = 0$, then none of the inputs is 1 and not all inputs are 0, and $l(\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n)) > 1 = l(\mathbf{x}_i)$ by Equations (1)—(3).

   We can now assume that $l(\mathbf{x}_i) > 1$, and thus that $\mathbf{x}_i$ has one of the following forms: $(01)^i$, $(01)^i 0$, $(10)^i$, or $(10)^i 1$, with $i > 0$. Also, by Part 1, we can assume that no side input is 1, and hence that each side input has at least one 0, that is $z(\mathbf{x}_j) > 0$ for all $j \in [n]$. Since

   $$\begin{aligned} z(\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n)) &= 1 + \sum_{h=1}^{n} (z(\mathbf{x}_h) - 1) \\ &= z(\mathbf{x}_i) + (z(\mathbf{x}_j) - 1) + \sum_{h=1, h \neq i,j}^{n} (z(\mathbf{x}_h) - 1), \end{aligned}$$

   we have

   $$z(\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n)) \geq z(\mathbf{x}_i) + (z(\mathbf{x}_j) - 1). \tag{14}$$

   Suppose there exists a $j \in [n]$ such that $z(\mathbf{x}_j) > 1$. Then $\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ has more 0s than $\mathbf{x}_i$, and any initial or final 1s of $\mathbf{x}_i$ must also appear in $\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n)$. Thus we have our result. There remain the words with one 0 to examine. The case where all the side inputs are 0 is covered by 2(d)–(g). Thus we may assume that at least one input, say $\mathbf{x}_j$, has length > 1.

(a) If $\mathbf{x}_i = (01)^i$, the only words with one 0 not considered in 2(d) are 10 and 101. Since $(01)^i \vee 10 = 1(01)^i$ and $(01)^i \vee 101 = 1(01)^i$, our claim follows.

(b) If $\mathbf{x}_i = (01)^i 0$, the only words with one 0 not considered in 2(e) are 01, 10 and 101. Since $(01)^i 0 \vee 01 = (01)^{i+1}$, $(01)^i 0 \vee 10 = (10)^{i+1}$, and $(01)^i 0 \vee 101 = 1(01)^{i+1}$, our claim follows.

(c) If $\mathbf{x}_i = (10)^i$, the only words with one 0 not considered in 2(f) are 01 and 101. Since $(10)^i \vee 01 = 1(01)^i$ and $(10)^i \vee 101 = 1(01)^i$, our claim follows.

(d) All the words with one 0 are considered in 2(g). □

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References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications* American Elsevier, 1976.

[2] J. A. Brzozowski and Z. Eśik, "Hazard Algebras," *Formal Methods in System Design,* **23(3)**, pp. 223–256, 2003.

[3] J. A. Brzozowski, Z. Ésik, and Y. Iland, "Algebras for hazard detection," *Beyond Two - Theory and Applications of Multiple-Valued Logic*, M. Fitting, and E. Orłowska, eds., pp. 3–24 (Physica-Verlag, 2003).

[4] J. A. Brzozowski and M. Gheorghiu, "Gate Circuits in the Algebra of Transients," *Theoretical Informatics and Applications*, **39**, pp. 67–91, 2005.

[5] J. A. Brzozowski and C-J. H. Seger, *Asynchronous Circuits*, Springer, 1995.

[6] E. B. Eichelberger, "Hazard detection in combinational and sequential switching circuits," *IBM J. Research and Development,* **9**, pp. 90–99, 1965.

[7] M. Gheorghiu and J. Brzozowski, "Simulation of feedback-free circuits in the algebra of transients," *Int. J. of Found. of Comp. Sci.,* **14(6)**, pp. 1033–1054, 2003.

[8] Y. Ye and J. A. Brzozowski, "Covering of Transient Simulation of Feedback-Free Circuits by Binary Analysis," *Int. J. of Found. of Comp. Sci.,* **17(4)**, pp. 949-973, 2006.