Circle Graph Obstructions

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis we present a self-contained proof of Bouchet's characterization of the class of circle graphs. The proof uses signed graphs and is analogous to Gerards' graphic proof of Tutte's excluded-minor characterization of the class of graphic matroids.

Acknowledgements

I am indebted to a number of individuals for which this thesis would not have been possible.

This thesis owes a lot to my family who have always been behind me every step of the way, and for which I would like to thank.

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Chapter 1

Introduction

A *circle graph* is the intersection graph of chords on a circle. De Frayessix [7] gave a natural correspondence between circle graphs and planar graphs; see Section 1.2. By his result, characterizations of the class of circle graphs can give rise to characterizations of the class of planar graphs.

Bouchet [1] characterized the class of circle graphs by a list of excluded vertex minors; see Figure 1.1 for that list, and see Section 1.1 for the definition of vertex minors. Not only is Bouchet's theorem analogous to Kuratowski's theorem for planar graphs, but via De Frayessix's theorem, one can derive Kuratowski's theorem as a consequence of Bouchet's characterization; see [9].

Theorem 1.0.1 (Bouchet [1]). *A simple graph is a circle graph if and only if it does not contain a vertex minor isomorphic to W_5, F_7, or W_7.*

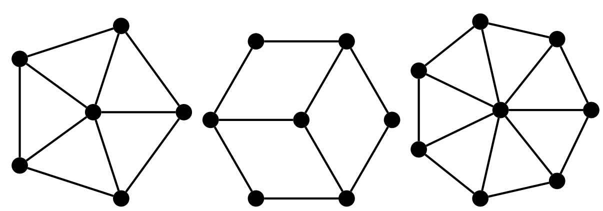

Figure 1.1: Excluded Vertex Minors – W_5, F_7, and W_7

Bouchet's original proof is long and relies on non-trivial connectivity results from isotropic systems. In this thesis we give a self-contained proof of Bouchet's Theorem using tools developed in graph theory since Bouchet's proof was first published in 1991. Our proof is inspired by Gerards' proof of Tutte's excluded minor characterization of the class of graphic matroids – see [10].

There has been considerable recent interest in vertex-minor closed classes of graphs. For example, Geelen recently conjectured that every proper vertex-minor closed class of graphs is chi-bounded; that is, the chromatic number of each graph is bounded above by a function of its clique number. This was known to hold for circle graphs; see Gyráfás in [11]. More recently, Dvořák and Král and Choi, Kwon, Oum, and Wollan have proved anagulous results for graphs with bounded rank-width and graphs excluding a wheel vertex-minor in [6] and [4] respectively. Vertex minors also arise in quantum computation – see Van den Nest, Dehaene, and De Moor in [19].

The class of circle graphs is believed to play a similar role for vertex minors that the class of planar graphs play for graph minors, which we discuss in Section 1.2.

1.1 Vertex Minors and Circle Graphs

Let G be a simple graph. For a vertex v, we let $N(v)$ denote the set of vertices adjacent to v in G. For a subset $S \subseteq V(G)$, we let $N(S) = (\cup_{v \in S} N(v)) \setminus S$; this is the set of vertices adjacent to S in G. The graph $G * v$ obtained by *locally complementing* at v is constructed from G by replacing $G[N(v)]$ in G with its complementary graph; see Figure 1.3 for an example. Two graphs G and H over the same vertex set are *locally equivalent* if one can be obtained from the other by some sequence of local complementations. We say that F is a *vertex minor* of G if F is isomorphic to some graph obtained from G by a sequence of local complementations and vertex deletions. Note that the order these operations are performed in matters. For example, in Figure 1.2, $G * 4 * 5 \setminus 4$ is the complete graph on four vertices, whereas $G * 5 * 4 \setminus 4$ is the graph on four isolated vertices; these two graphs are not locally equivalent.

A *chord diagram* $\mathcal{C}$ is a drawing of a unit circle and some labeled straight-line chords $C(\mathcal{C})$ with disjoint ends on the unit circle in $\mathbb{R}^2$. An *arc* of a chord diagram is an arc of the circle between the ends of two chords that does not intersect a third. We view chords as labeled subsets of $\mathbb{R}^2$. Given a chord diagram, its *intersection graph* $IG(\mathcal{C})$ is a graph $(C(\mathcal{C}), E(\mathcal{C}))$ where $E(\mathcal{C}) = \{\{c, d\} : c, d \in C(\mathcal{C}), c \cap d \neq \varnothing\}$. A *circle graph* is the intersection graph of the chords of some chord diagram; see Figure 1.4 for an example.

The intersection structure of a chord diagram can be recovered from the order its chords appear on a clockwise walk on a circle. Note that two chords x and y of a chord diagram

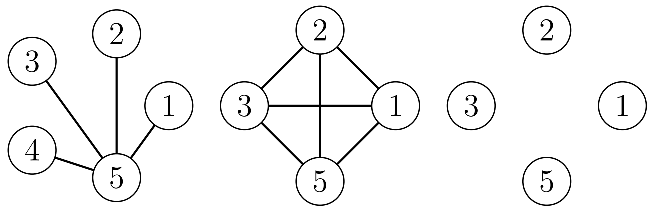

Figure 1.2: G, $G * 4 * 5 \setminus 4$, and $G * 5 * 4 \setminus 4$.

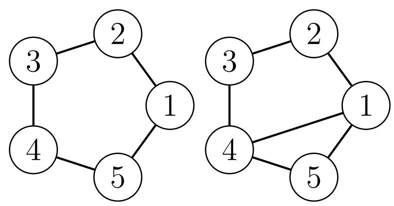

Figure 1.3: Two locally equivalent graphs: G and $G * 5$

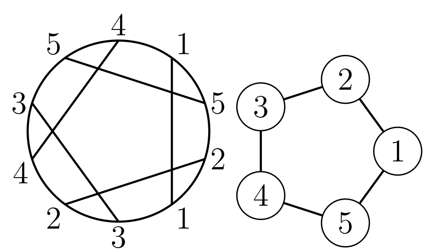

Figure 1.4: A Chord Diagram and its Circle Graph

$\mathcal{C}$ cross if and only if they appear in interlaced order on a walk around $\mathcal{C}$ – that is, first x then y then x then y.

This allows us to define an alternate representation for chord diagrams. From every chord diagram $\mathcal{C}$, we can obtain a *double occurrence word* T of labels of $\mathcal{C}$ by writing down the labels of chords of $\mathcal{C}$ as their ends are encountered starting from an arbitrary point on the circle. Two double occurrence words are *equivalent* if they are equal modulo string reversal and rotation, that is:

$$(w_1, w_2, \dots w_n) \text{ is equivalent to } (w_n, w_{n-1} \dots w_1), \text{ and}$$
$$(w_1, w_2 \dots w_n) \text{ is equivalent to } (w_2, w_3 \dots w_n, w_1).$$

Note that the set of all possible double occurrence words that can be obtained from a chord diagram via this construction forms an equivalence class of double occurrence words. Furthermore, from a chord diagram and a double occurrence word for it, we can recover a chord diagram that shares the same intersection structure as the original by writing down the double occurrence word around a circle and drawing chords between equal labels. As we are only concerned about the combinatorial intersection structure of a chord diagram in this thesis, we say two chord diagrams are *equivalent* if they share the same set of double occurrence words.

Let c be a chord in a chord diagram $\mathcal{C}$. The chord diagram $\mathcal{C} * c$ is constructed from $\mathcal{C}$ by reversing the order in which chords appear in $\mathcal{C}$ on one side of c; see Figure 1.5 for an example. Note that this operation does not affect any chords that did not cross c, and

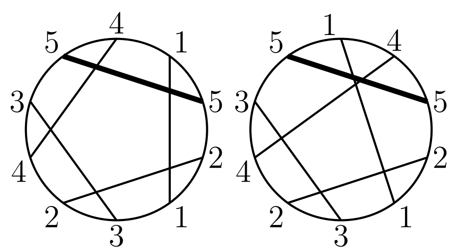

Figure 1.5: $\mathcal{C}$ and $\mathcal{C} * 5$

for two chords c_1 and c_2 that cross c, we have that c_1 and c_2 cross in $\mathcal{C}$ if and only if they do not in $\mathcal{C} * c$, which corresponds to local complementation in the circle graph given by IG($\mathcal{C}$). This result was first observed by Kotzig.

Lemma 1.1.1 (Kotzig [14])**.** *$IG(\mathcal{C}) * c = IG(\mathcal{C} * c)$.*

Hence the class of circle graphs is closed under local complementation, and thus vertex minors.

1.2 Circle Graphs and Planar Graphs

To illustrate the connection between circle graphs and planar graphs, we must first go through a construction known as the *fundamental graph*. The *fundamental graph* $F(G, T)$ of a graph G relative to a spanning forest T of G is the bipartite graph over the edges of G given by the bipartition $(E(T), E(G) \setminus E(T))$ where an edge $t \in E(T)$ is adjacent to an

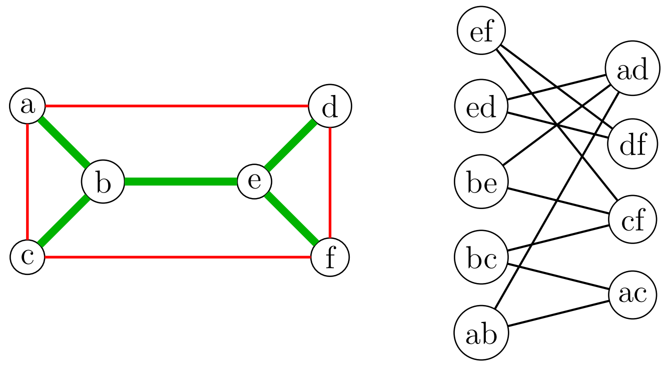

Figure 1.6: G and $F(G,T)$; T given by thick green edges.

edge $e \in E(G) \setminus E(T)$ in $F(G,T)$ if and only if the fundamental cycle of e in T contains t. Figure 1.6 gives an example of a graph G and its fundamental graph $F(G,T)$ for a chosen spanning tree T.

De Frayessix [7] gave a natural correspondence between fundamental graphs of plane graphs and bipartite chord diagrams. Starting from a connected planar graph G already embedded in the plane, and a spanning tree T of G, one may construct a chord diagram by:

1. drawing a simple closed circle "conforming" to T,
2. replacing T with "perpendicular" chords, and finally by
3. flipping the edges in $E(G) \setminus T$ into the interior of the circle,

as illustrated in Figure 1.7. Note that the chords corresponding to the spanning tree edges only cross the chords corresponding to non-spanning tree edges if and only if the spanning tree edge is in the fundamental cycle of the non-spanning tree edge. This proves the following result of De Frayessix.

Theorem 1.2.1 (De Frayessix [7]). *Let G be a simple bipartite graph. Then G is a circle graph if and only if G is a fundamental graph of some planar graph.*

One can also obtain the following result characterizing circle graphs through De Frayessix's result characterizing bipartite circle graphs.

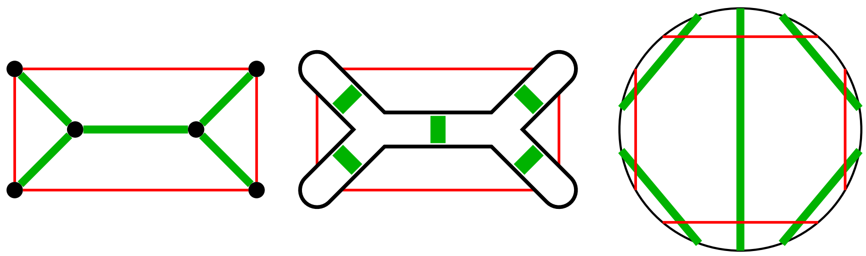

Figure 1.7: Planar Graph to a Bipartite Chord Diagram – An Illustration

Theorem 1.2.2 (Folklore)**.** *Let G be a simple graph. Then G is a circle graph if and only if G is a vertex minor of a fundamental graph of some planar graph.*

Fundamental graphs also illustrate an interesting connection between minors of graphs and vertex minors. For example, for any graph G, the fundamental graph $F(G/t, T/t)$ is isomorphic to $F(G,T) \setminus t$. Similarly, the fundamental graph $F(G, T\Delta\{t,e\})$ for a spanning tree edge t and an edge e whose fundamental cycle contains t is given by $F(G,T) * t * e * t$. By these two results one can obtain the following result connecting minors and vertex minors.

Lemma 1.2.3 (Bouchet)**.** *Let F and G be two graphs, and let T_F and T_G be spanning forests for F and G respectively. Then F is a minor of G if and only if $F(F, T_F)$ is a vertex minor of $F(G, T_G)$.*

As these results are not important for our proof of Bouchet's Theorem we will omit proofs for them.

1.3 Four Regular Graphs

Vertex minors are not as easy to work with as minors; as seen previously, local complementations and vertex deletions do not necessarily commute. Fortunately, we can encode a local equivalence class of circle graphs as the set of Eulerian tours of an associated connected four-regular graph R. Let $\mathcal{C}$ be any chord diagram representing a circle graph G. From $\mathcal{C}$ we may construct a connected four-regular graph R; take the chords of $\mathcal{C}$ to be the vertices of R and the arcs of $\mathcal{C}$ to be the edges of R, and T to be a double occurrence word for $\mathcal{C}$. Note that T gives an Eulerian tour of R as the arcs of $\mathcal{C}$ are the edges of R. For example, in Figure 1.8, the canonical tour would be given by the double occurrence word $(5, 4, 1, 3, 2, 1, 2, 5, 4, 3)$. We call R the *tour graph* associated with $\mathcal{C}$, and we will use $R(\mathcal{C})$ to refer to it. Note that T completely determines R. We say that T is a *tour* of R.

Conversely, a tour T of a four-regular graph R gives rise to a double occurrence word, as every vertex is visited twice in T. Hence we may obtain a chord diagram from a tour T of a four regular graph R.

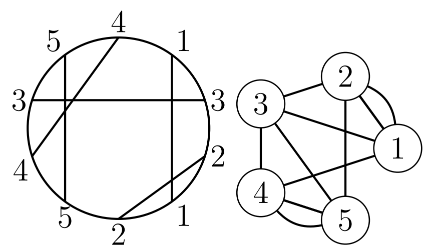

Figure 1.8: A Chord Diagram $\mathcal{C}$ with Associated Four-Regular Graph R.

Locally complementing at a vertex v in G is equivalent to switching the transition of our tour at the vertex v in R. For example, in $C * 5$, shown in Figure 1.9, the canonical Eulerian tour would be given by (5, 4, 1, 3, 2, 1, 2, 5, **3**, **4**). Hence locally equivalent chord diagrams correspond to different Euler tours of the same underlying tour graph R.

Conversely, any two tours of a given connected four-regular graph R correspond to locally equivalent circle graphs.

Lemma 1.3.1 (Kotzig [13])**.** *Let T_1 and T_2 be two tours of a connected four-regular graph R. Then the corresponding chord diagrams $\mathcal{C}_1$ and $\mathcal{C}_2$ are locally equivalent.*

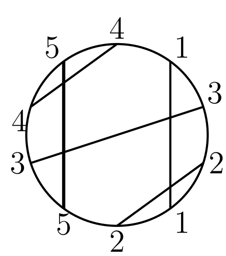

Figure 1.9: $\mathcal{C} * 5$

Proof. Proceed by induction on the number of transitions for which T_1 and T_2 disagree. If there are no disagreements, then $T_1 = T_2$ and hence $\mathcal{C}_1 = \mathcal{C}_2$, as desired.

Otherwise, there is a vertex v for which T_1 and T_2 disagree. Let T_1' be the tour for $C_1 * v$ and T_2' be the tour for $C_2 * v$. Now as R is four-regular, there are only three possible transitions at v in R. Hence one of T_1 or T_1' is equal to one of T_2 or T_2'. By induction, the corresponding chord diagrams are locally equivalent and hence $\mathcal{C}_1$ is locally equivalent to $\mathcal{C}_2$, as desired. □

Henceforth we will say a *tour graph* R is a connected four-regular graph. Note that tour graphs need not be simple; the tour graph for the chord diagram on a single isolated chord is a vertex is a vertex incident to two loops.

Deletion in a circle graph also gives rise to a notion of vertex removal in a tour graph R. For a chord $c \in C(\mathcal{C})$, to *split off* at c is to remove c and to identify the (two) pairs of ends of edges of R incident to c. Whenever we would identify the two ends of a loop together we simply delete the loop. Observe that there are only three ways to pair up the four ends of edges incident to c; one can show that these correspond to the three ways that a vertex can be removed up to local equivalence in any given graph G. For two four-regular graphs S and R, we say that S is an *immersion minor* of R if S is isomorphic to a graph obtained from R by splitting off vertices of R. With the above observation on splitting off vertices, this notion captures the vertex-minor relation between two circle graphs; given an circle graph G with tour graph R_G, and an circle graph H with tour graph R_H, if R_H is an immersion minor of R_G, then H is a vertex-minor of G.

1.4 Prime Decompositions

A circle graph may have multiple inequivalent chord diagram representations; this should not be too surprising, as we have a correspondence between bipartite chord diagrams and fundamental graphs of plane graphs, and planar graphs may have inequivalent plane embeddings. This is awkward as in a proof of Bouchet's Theorem (or any proof of an excluded-minors theorem), one would like to characterize those vertex minor-minimal graphs F for which a chord diagram representation of $F \setminus v$ cannot be extended to a chord diagram for F. The difficulty lies in the fact that $F \setminus v$ may have multiple inequivalent representations, so we may need to analyze them all. When one proves Kuratowski's Theorem, one solves this problem by proving three lemmas; one which states that simple 3-connected planar graphs have only one plane embedding up to homotopy in the sphere, Tutte's wheels theorem for decomposing 3-connected planar graphs, and one which states minor-minimal

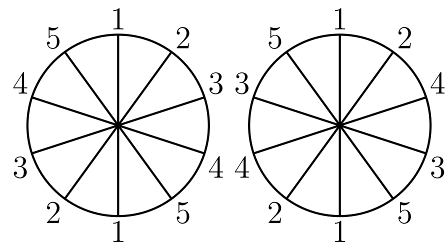

Figure 1.10: Inequivalent Chord Diagrams

non-planar graphs must be 3-connected and simple. In this section we introduce similar machinery for circle graphs and vertex minors.

For a subset of vertices A of a graph G, we say the *edge cut* defined by A, to be $\delta(A) = \{e \in E(G) : e \text{ is incident to a vertex in } A\} \setminus E(G[A])$. A *split* of a graph G is a bipartition (A, B) of $V(G)$ where $G[\delta(A)]$ is a complete bipartite graph and $|A| \geq 2$ and $|B| \geq 2$; see Cunningham [5]. We say a graph G is *prime* if it contains no splits.

The following result of Cunningham illustrates the connection between splits and 3-connectivity. As we will not use this result, we will not prove it.

Lemma 1.4.1 (Cunningham [5])**.** *Let G be a graph and $F(G,T)$ be a fundamental graph for G. Then G is simple and 3-connected if and only $F(G,T)$ is prime.*

Splits, like two-separations in planar graphs, pose an obstruction for unique representability. This is best illustrated by an example; Figure 1.10 illustrates two chord diagrams for the circle graph K_5. Note that K_5 has many splits; for instance, the bipartition $(\{1,2,5\},\{3,4\})$. However, these diagrams are inequivalent as they give different labeled tour graphs, as shown in Figure 1.11. The problem is that we can flip the order in which 3 and 4 appeared on the chord diagram and in the tour graph, as $(\{1,2,5\},\{3,4\})$ was a split. This is similar to the the problem two-separations pose in planar graphs, in which a face can be "flipped" to be embedded on one side or on the other side of a two-separation.

Fortunately splits are the only obstruction to unique representability, and vertex-minor-minimal non-circle graphs are split-free. To see this, we will introduce some more notation. The graph $G \downarrow X$, for $X \subseteq V(G)$, is formed by identifying X down to a single vertex; formally we construct it from G by deleting X and adding a new vertex that is adjacent to the vertices in $N(X) \cap Y$; this is illustrated in Figure 1.12. Circle graphs decompose over splits; from a chord diagram for $G \downarrow Y$ and a chord diagram for $G \downarrow X$ we may construct a

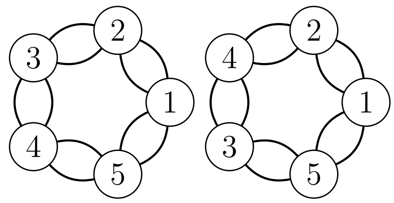

Figure 1.11: Inequivalent Tour Graphs

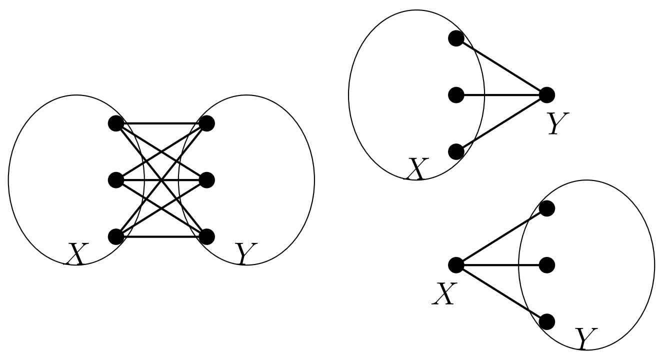

Figure 1.12: G, $G \downarrow Y$, and $G \downarrow X$

chord diagram for G by overlaying the two diagrams; the proof of this result is illustrated in Figure 1.13.

Lemma 1.4.2 (Bouchet [1], Naji [15], and Gabor, Hsu, and Supowit [8])**.** *Let G be a graph with a split (X, Y). Then G is a circle graph if and only if both $G \downarrow X$ and $G \downarrow Y$ are circle graphs.*

Since $G \downarrow X$ and $G \downarrow Y$ are both isomorphic to induced subgraphs of G for a split (X, Y), excluded vertex-minors of the class of circle graphs are prime. One can show that circle graphs that are prime have exactly one chord diagram up to equivalence; the proof is a straightforward inductive argument using Lemma 1.4.5.

Lemma 1.4.3 (Bouchet [1], Naji [15], and Gabor, Hsu, and Supowit [8])**.** *Let G be a circle graph. If G is prime, then G has an unique chord diagram $\mathcal{C}$ representing it.*

We also obtain decomposition tools for the tour graph R, via the following observation. A tour graph R is *internally six-edge connected* if it is four edge connected and any four-edge cut splits R into at most two connected components, where one side has at most one

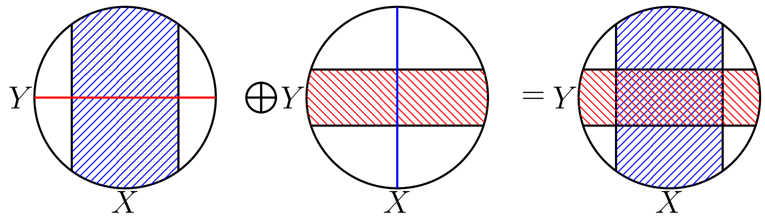

Figure 1.13: Chord diagrams for $G \downarrow Y$, $G \downarrow X$ and G

vertex. As R is Eulerian, there are no five-edge cuts. We prove the following correspondence between prime circle graphs and internally six-edge connected four regular graphs in Chapter 4. A *tour graph* for a circle graph G is a tour graph for a chord diagram for G.

Lemma 1.4.4 (Bouchet [1])**.** *Let G be a circle graph with tour graph R. Then G is prime if and only if R is internally six-edge connected.*

The following result is akin to Tutte's Wheels Theorem for 3-connected simple graphs; we prove this in Chapter 4.

Lemma 1.4.5 (Bouchet [3])**.** *Let G be a prime graph. Either G is locally equivalent to C_5 or there is a graph G' locally equivalent to G such that $G' \setminus v$ is prime for some vertex $v \in V(G)$.*

1.5 Extended Representations

By Lemmas 1.4.2 and 1.4.5 we know that the vertex-minor-minimal non-circle graphs are prime, and there is a vertex which we can remove to obtain a prime circle graph. In light of this fact we would like a succinct way to describe single vertex extensions of circle graphs; to this end we introduce hyperchords. A *hyperchord* Σ for a chord diagram $\mathcal{C}$ is an even subset of the arcs of $\mathcal{C}$. Every chord in a chord diagram $\mathcal{C}$ partitions the set of arcs into two parts – those on one side of c in $\mathcal{C}$ and those on the other. Hence every chord partitions a hyperchord Σ into two parts. We say a hyperchord Σ crosses a chord c of $\mathcal{C}$ if the partition c induces in Σ consists of two parts of odd size. An arc is *odd* if it is in Σ, even otherwise.

This notion of a hyperchord is a natural generalization of that of a chord; note that when $|\Sigma| = 2$, a hyperchord Σ can be replaced with a simple chord crossing the same set of chords as Σ, by drawing a new chord with one end in one odd arc and the other end in the other odd arc. From an extended chord diagram $(\mathcal{C}, \Sigma)$ we obtain an *extended circle graph* $\mathrm{IG}(\mathcal{C}, \Sigma)$, by adding a new vertex v for the hyperchord Σ to $\mathrm{IG}(\mathcal{C})$ where v is adjacent to c if and only if c crosses Σ. As it turns out, this construction is rather useful, as it captures the structure of single-vertex extensions of circle graphs; we give a proof of this result in Chapter 5.

Lemma 1.5.1 (Bouchet [1])**.** *Every single-vertex extension of a circle graph is an extended circle graph.*

Note that deletion carries through to extended circle graphs; when deleting a chord, simply merge adjacent arcs preserving parity. Likewise, this notion of a extended chord diagram behaves well with respect to local complementation, so long as it is not the extension vertex being locally complemented.

Lemma 1.5.2. *Let* $H = \mathrm{IG}(\mathcal{C}, \Sigma)$*, and let* c *be a chord of* $\mathcal{C}$*. Then* $H * c = \mathrm{IG}(\mathcal{C} * c, \Sigma)$*.*

We would like to relate the structure a hyperchord Σ has in relation to a chord diagram $\mathcal{C}$ to the underlying tour graph R for $\mathcal{C}$. As before, an extended chord diagram is hard to work with; we would like a succinct way to describe the combinatorial structure of an extended chord diagram. First note that as Σ is a subset of the arcs of $\mathcal{C}$, Σ is a subset of the edges of R, as the edges of R are the arcs of $\mathcal{C}$.

Futhermore, two hyperchords Σ_1 and Σ_2 over $\mathcal{C}$ which give rise to the same extended circle graph G are related by cuts of the tour graph R. Again, we give a proof of this result in Chapter 5.

Lemma 1.5.3 (Bouchet [1])**.** *Let* Σ_1 *and* Σ_2 *be two hyperchords of some chord diagram* $\mathcal{C}$*. If* $\mathrm{IG}(\mathcal{C}, \Sigma_1) = \mathrm{IG}(\mathcal{C}, \Sigma_2)$*, then* $\Sigma_1 \Delta \Sigma_2$ *is a cut of* $R(\mathcal{C})$*. Morever, for any cut* X *of* $R(\mathcal{C})$*,* $\mathrm{IG}(\mathcal{C}, \Sigma_1) = \mathrm{IG}(\mathcal{C}, \Sigma_1 \Delta X)$*.*

This combinatorial structure is exactly that described by a *signed graph* over R. A *signed graph* (G, Σ) is a graph G equipped with a special subset of edges $\Sigma \subseteq E(G)$; two signed graphs (G_1, Σ_1) and (G_2, Σ_2) are equivalent if $G_1 = G_2$ and $\Sigma_1 \Delta \Sigma_2$ is a cut of G_1. A *signature* of a signed graph (G, Σ) is any set Σ' for which $\Sigma' \Delta \Sigma$ is a cut. An edge is *odd* if it is in Σ, and even otherwise. A *signed tour graph* is a signed graph over a tour graph with an even-sized signature. As cuts of a tour graph have even size, this is well defined.

By this there is a correspondence between extended circle graphs $\mathrm{IG}(\mathcal{C}, \Sigma)$ and signed tour graphs with an Eulerian tour (R, Σ, T); Σ is simply the hyperchord that we extend $\mathcal{C}$ by, and T is the tour to take on R to get $\mathcal{C}$. Henceforth we say $\mathrm{IG}(R, \Sigma, T)$ is the extended

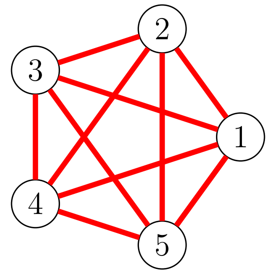

Figure 1.14: Even Signed Tour Graph $(K_5, E(K_5))$ – Thick Red Edges are Odd.

circle graph given by $\mathrm{IG}(\mathcal{C}, \Sigma)$ where $\mathcal{C}$ is the chord diagram given by (R, T). An example is illustrative here; Figure 1.14 depicts the even signed tour graph $(K_5, E(K_5))$. Taken with the tour $T = (5, 4, 1, 5, 2, 1, 3, 2, 4, 5)$, one obtains the extended chord diagram $(\mathcal{C}, E(K_5))$, where every arc of $\mathcal{C}$ is odd; $\mathcal{C}$ is shown in Figure 1.15. As every chord crosses the hyperchord, the resulting extended circle graph $\mathrm{IG}(R, \Sigma, T) = \mathrm{IG}(\mathcal{C}, \Sigma)$ is the graph with a 5-cycle and an additional vertex adjacent to every vertex on the 5-cycle; namely the 5-wheel W_5. Note that W_5 is one of the three obstructions in Bouchet's Theorem; the other two obstructions, W_7 and F_7, have representations as signed tour graphs depicted in Figures 1.16 and 1.17 respectively.

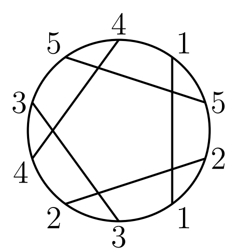

Figure 1.15: A Chord Diagram

Lemmas 1.5.2 and 1.5.3, restated in terms of even signed tour graphs, state the following.

Lemma 1.5.4 (Bouchet [1]). *Let* $H = \mathrm{IG}(R, \Sigma, T)$*, and let* c *be a vertex of* R*. Then* $H * c = \mathrm{IG}(R, \Sigma, T')$*, for some Eulerian tour* T' *of* R*. Conversely, for two Eulerian tours* T *and* T' *of* R*,* $\mathrm{IG}(R, \Sigma, T)$ *and* $\mathrm{IG}(R, \Sigma, T')$ *are locally equivalent.*

Lemma 1.5.5 (Bouchet [1]). *Let* Σ_1 *and* Σ_2 *be two even sized subsets of edges of a tour graph* R *with tour* T*. If* $\mathrm{IG}(R, \Sigma_1, T) = \mathrm{IG}(R, \Sigma_2, T)$*, then the two signed graphs* (R, Σ_1)

and (R, Σ_2) *are equivalent. Moreover, for two equivalent signed graphs* (R, Σ_1) *and* (R, Σ_2)*,* $\mathrm{IG}(R, \Sigma_1, T) = \mathrm{IG}(R, \Sigma_2, T)$*.*

Now by Lemma 1.5.5, an extended circle graph $\mathrm{IG}(R, \Sigma, T)$ is a circle graph if there is a cut X of $R(\mathcal{C})$ such that $\Sigma \Delta X$ has size at most two, which is equivalent to saying (R, Σ) has a signature Σ' of size two or less. Unfortunately the converse is not always true; there is a difficulty here in that a circle graph can have inequivalent representations, and hence inequivalent tour graph representations for which one which may admit an extension but the other may not. We sidestep this difficulty by restricting ourselves to only considering extensions of circle graphs with unique representations; note that prime circle graphs have unique representations, and by Lemma 1.4.5, an excluded vertex-minor for the class of circle graphs admits a representation as an extension of a prime circle graph.

Lemma 1.5.6 (Bouchet [1])**.** *Let* (R, Σ) *be a internally six-edge connected even signed tour graph. Then* $\mathrm{IG}(R, \Sigma, T)$ *is a circle graph if and only if* (R, Σ) *has a signature of size* 0 *or* 2*.*

We give proofs for all three of the above lemmas in Chapter 5.

1.6 A Key Lemma

With Lemma 1.5.6 in mind, we would like to characterize those four-regular signed graphs which cannot be resigned to signatures containing only two edges.

A cycle of (R, Σ) is *odd* if it has an odd number of odd edges. Every odd cycle of (R, Σ) will intersect every signature of (R, Σ) in an odd number of odd edges, as cuts intersect cycles in even parity. Hence one obstruction is having three edge-disjoint odd cycles, as each odd cycle has at least one odd edge. Another obstruction is the graph odd-K_5, which is the signed graph given by $(K_5, E(K_5))$ – any equivalent signed graph will have at least four odd edges.

Both splits and immersions lift up to signed four-regular tour graphs; when we identify two edges together to get an edge in the immersion minor, we also preserve parity. Two odd edges and two even edges will be identified to a single even edge, and an odd edge and an even edge will be identified to an odd edge in the immersion minor. Note that given two extended circle graphs $G = \mathrm{IG}(R_G, \Sigma_G, T_G)$ and $H = \mathrm{IG}(R_H, \Sigma_H, T_H)$ if (R_H, Σ_H) is an immersion minor of (R_G, Σ_G), then H is a vertex-minor of G.

With these two ideas in mind, we prove the following new lemma, which is a key step towards our proof of Bouchet's Theorem.

Lemma 1.6.1. *Let* (R, Σ) *be a loopless signed tour graph. Then either:*

- *There is a cut* C *of* R *such that* $|\Sigma \Delta C| \leq 2$*, or*
- (R, Σ) *has* 3 *edge-disjoint odd circuits, or*
- (R, Σ) *has an odd-*K_5 *immersion minor.*

1.7 Bouchet's Theorem

By Lemmas 1.4.2 and 1.5.1 we have that a minimal non-circle graph G admits a representation as a signed internally-six-edge connected tour graph with tour (R, Σ, T). By Lemma 1.6.1 we have that either (R, Σ) packs three edge-disjoint odd circuits or it has an odd-K_5-immersion minor. We have already seen above that odd-K_5 with the appropriate tour is a representation of the 5-wheel W_5, so we are done in this case.

Otherwise, (R, Σ) admits a packing of three edge-disjoint odd circuits C_1, C_2, and C_3. Eulerian graphs admit a decomposition into edge-disjoint circuits; in particular, $R \setminus (C_1 \cup C_2 \cup C_3)$ is Eulerian. As we deleted an odd number of odd edges, there are still an odd number of odd edges remaining, so there is at least one more odd circuit remaining in $R \setminus (C_1 \cup C_2 \cup C_3)$ Hence three edge-disjoint odd circuits give rise to four for free, so R admits a packing of four edge-disjoint odd circuits $\mathcal{P}$. As every proper vertex minor of G is a circle graph, (R, Σ) is immersion-minor-minimal with respect to being both internally six-edge connected and having a packing of four-edge disjoint odd circuits.

We prove this new key lemma characterizing immersion-minor-minimal internally six-edge-connected graphs packing four edge-disjoint odd circuits.

Lemma 1.7.1. *If* (R, Σ) *is an signed immersion-minor-minimal internally-six-edge-connected tour graph packing four edge-disjoint odd circuits with then*

- $|V(R)| = 6$ *and* (R, Σ) *is equivalent to* $R(W_7)$*, as shown in Figure 1.16, or*
- $|V(R)| = 7$ *and* (R, Σ) *is equivalent to* $R(F_7)$*, as shown in Figure 1.17.*

Bouchet's Theorem then follows as a direct consequence of Lemmas 1.7.1 and 1.6.1; by these Lemmas, if (R, Σ) represents a minimal non-circle graph without an odd-K_5-immersion minor, then (R, Σ) can be resigned to $R(F_7)$ or $R(W_7)$. Now, $R(F_7)$ with the tour $(1, 6, 2, 1, 3, 2, 1, 3, 5, 4, 6, 5, 1)$ gives a representation for F_7, and $R(W_7)$ with the

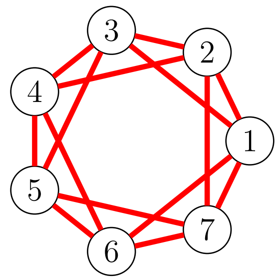

Figure 1.16: Signed Graph $R(W_7)$ – Thick Red Edges are Odd.

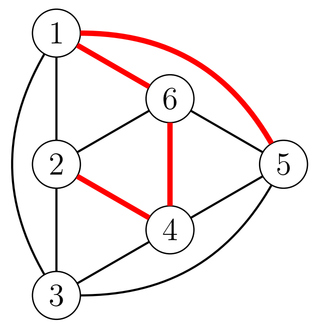

Figure 1.17: Signed Graph $R(F_7)$ – Thick Red Edges are Odd.

tour $(1, 7, 2, 1, 3, 2, 4, 3, 5, 4, 6, 5, 7, 6, 1)$ gives a representation for W_7. Hence we have the vertex-minor minimal non-circle graphs are locally equivalent to one of W_5, W_7, or F_7, as desired.

Chapter 2

Signed Tour Graphs

We start by proving our two new results characterizing signed tour graphs that cannot be resigned to a signature with at most two odd edges. Recall that a *signed tour graph* is a even signed graph (R, Σ) where R is a tour graph. Recall that a *tour graph* R is a four-regular connected graph.

The following two lemmas are useful when working with four-regular signed graphs.

Lemma 2.0.1. *Let (R, Σ) be a signed tour graph, and let (R', Σ') be an immersion minor of (R, Σ). If (R', Σ') has at least c edge-disjoint odd circuits, then so does (R, Σ).*

Proof. Let $\mathcal{P}'$ be a packing of c edge-disjoint odd circuits of (R', Σ'). Now the preimage of a signed graph under a split is obtained by subdividing two edges preserving parity followed by identifying the resulting vertices. Hence the preimage of an edge-disjoint odd-circuit packing is an edge-disjoint odd circuit packing, as desired. □

Lemma 2.0.2. *Let v be a vertex in a four-regular signed graph (R, Σ). If every signature of (R, Σ) has at least n odd edges, then any graph obtained by splitting off at v has at least $n - 2$ edges in its signature.*

Proof. We may assume without loss of generality that v has at most two odd edges incident to it in (R, Σ). Let (R', Σ') be the resulting graph obtained by splitting off at v in (R, Σ). Let e and f be the resulting identified edges. Now (R, Σ) can be obtained by (R', Σ') by subdividing e and f, possibly resigning at one of the two new vertices, and identifying the two new vertices. This adds at most two edges to the signature, as desired. □

2.1 Finding Odd-K_5

Recall that *odd-K_5* is the signed graph $(K_5, E(K_5))$. A *balanced* subgraph H of a four-regular signed graph (R, Σ) is a subgraph such that H does not contain any odd circuit of (R, Σ). Note that balanced subgraphs are invariant under resigning and that there is a resigning of (R, Σ) to (R, Σ') such that $|E[H] \cap \Sigma'| = 0$; see [12] for a proof. A *1-separation* of a graph G is a partition of the vertex set of G into two parts (A, B) such that $|A \cap B| = 1$. The *common vertex* of a 1-separation (A, B) is the one vertex in $A \cap B$. For two disjoint subsets of vertices A and B of a graph R, we say that $\delta(A, B)$ is the set of edges linking A and B; namely, $\delta(A, B) = \{\{a, b\} \in E(R) : a \in A, b \in B\}$.

We obtain the following lemma:

Lemma 2.1.1. *Let (R, Σ) be a loopless four-regular signed graph with at least α odd edges in every signature, with $\alpha \leq 4$, and $\alpha \equiv |\Sigma| \pmod 2$. Then either:*

- *(R, Σ) has α edge-disjoint odd circuits, or*
- *(R, Σ) has an odd-K_5 as an immersion minor.*

Proof. Suppose not for a contradiction. Let (R, Σ) be a counterexample with $|V(R)|$ minimal. By repeatedly splitting off vertices with Lemma 2.0.2 we may assume without loss of generality that $|\Sigma| \leq 4$. As (R, Σ) is a counterexample we may assume that $|\Sigma| = \alpha$. As an Eulerian graph with a single odd edge contains an odd closed walk, namely the tour itself, and hence an odd circuit, we may assume without loss of generality $\alpha > 1$.

Claim 2.1.2. *There is no partition of $V(R)$ into (A, B) with both $|\delta(A)| = 2$ and $R[A]$ balanced.*

Proof. Suppose not for a contradiction. Resign (R, Σ) to (R, Σ') such that $|E(R[A]) \cap \Sigma'| = 0$. Let $F = \{e, f\} = \delta(A)$. We may assume without loss of generality that $F \nsubseteq \Sigma'$ by resigning through F. We may also assume that $f \notin \Sigma'$, again by resigning through F if necessary. As R admits an Eulerian tour, we may split apart the vertices in $R[A]$ following the transition the tour uses to identify $R[A] \cup F$ to e. As splitting preserves the parity of edges, and as f was not an odd edge, we have that the resulting signed graph is given by (R', Σ'). As (R', Σ') is an immersion minor of (R, Σ'), it cannot have odd-K_5 as an immersion minor. Now as any cut of R' lifts to a cut of R with the same cardinality, we have that every signature of (R', Σ') lifts to a signature of (R, Σ) with the same size. Hence, by minimality, every signature of (R', Σ') has at least α many odd edges. Therefore

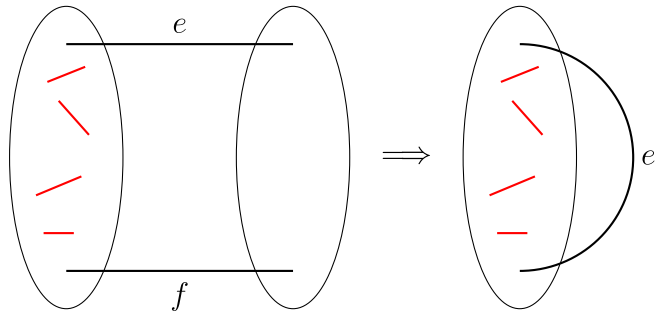

Figure 2.1: Balanced Two-Edge Cut

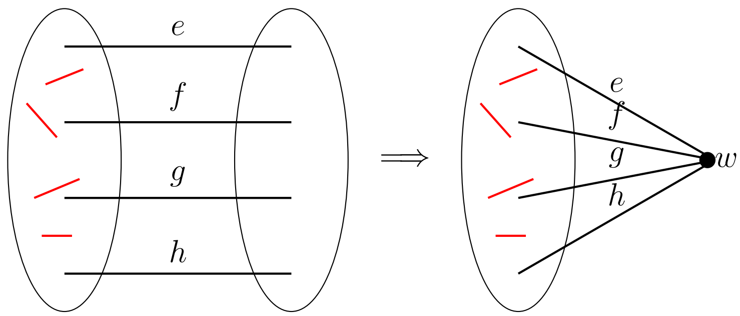

Figure 2.2: Balanced Four-Edge Cut

(R', Σ') has α edge-disjoint odd circuits, and therefore so does (R, Σ) by Lemma 2.0.1, a contradiction, as desired. $\square$

Claim 2.1.3. *There is no partition of $V(R)$ into (A, B) with both $|A| > 1$, $|B| > 1$, $|\delta(A)| = 4$ and $R[A]$ balanced.*

Proof. Suppose not for a contradiction. Resign (R, Σ) to (R, Σ') such that $|E(R[A]) \cap \Sigma'| = 0$. As $R[A]$ is balanced, by Claim 2.1.2 we have there is no two-edge cut in $R[A]$. Let $\{e, f, g, h\} = \delta(A)$, and let w be some vertex in $R[A]$. By Menger's Theorem there are four-edge disjoint paths from w to A. We may identify these paths by the edge they use in $\delta(A)$ as $|\delta(A)| = 4$; hence let P_e, P_f, P_g, and P_h be four-edge disjoint paths from w to A, using e, f, g, and h respectively. Now split off the other vertices in R to obtain a new signed tour graph (R', Σ') such that w is incident to e, f, g, and h; we can do so by splitting off the vertices in each of P_e, P_f, P_g and P_h in a way that preserves the paths P_e, P_f, P_g, and P_h.

As (R', Σ') is an immersion minor of (R, Σ'), it cannot have odd-K_5 as an immersion minor. Now as any cut of R' lifts to a cut of R with the same cardinality and parity, we have that every signature of (R', Σ') lifts to a signature of (R, Σ) with the same size. Hence we have that every signature of (R', Σ') has at least α many odd edges. Therefore (R', Σ') has α edge-disjoint odd circuits, and therefore so does (R, Σ) by Lemma 2.0.1, a contradiction, as desired. □

Claim 2.1.4. *R is 2-connected.*

Proof. Suppose not for a contradiction. Let (A_1, A_2) be a 1-separation with common vertex v. Let $R_1 = R[A_1]$, and let $R_2 = R[A_2]$. Let $\Sigma_1 = \Sigma \cap E(R_1)$, and let $\Sigma_2 = \Sigma \cap E(R_2)$. By resigning at v we may assume that $|\Sigma_1 \cap \delta(v)| \leq 1$ and $|\Sigma_2 \cap \delta(v)| \leq 1$. Now consider the signed graphs (R_1, Σ_1) and (R_2, Σ_2), and let (R'_i, Σ_i) be the graph obtained by unsubdividing the edge split by v while preserving parity; note that v has degree two in R_i for $i \in \{1, 2\}$. Let α_1 be the minimum size of a signature for (R'_1, Σ'_1) and let α_2 be the minimum size of a signature for (R'_2, Σ'_2). Let Σ''_1 and Σ''_2 be signatures realizing these sizes. By minimality neither (R'_1, Σ'_1) nor (R'_2, Σ'_2) has an odd-K_5-immersion minor, as otherwise so would (R, Σ). Hence we have that (R'_1, Σ'_1) has α_1 many edge-disjoint odd circuits and (R'_2, Σ'_2) has α_2 many edge-disjoint odd circuits, with $\alpha_1 + \alpha_2 < \alpha$, as (R, Σ) has less than α many edge-disjoint odd circuits. However, since every cut of (R'_1, Σ'_1) and (R'_2, Σ'_2) lifts to a cut with the same cardinality and parity to a cut of (R, Σ), $\Sigma''_1 \cup \Sigma''_2$ lifts to a signature Σ' of (R, Σ) with size $\alpha_1 + \alpha_2$. However, as the minimum size of a signature of (R, Σ) was α, which was strictly larger than $\alpha_1 + \alpha_2$, this is a contradiction, as desired. □

Two cases follow: either every vertex is incident with a parallel pair or there is some $v \in V(R)$ that is not incident with a parallel pair.

Case 1: *Every vertex is incident with a parallel pair.* If there is a vertex incident to a parallel quadruple of edges, since R is connected and four-regular, we have that $|V(R)| = 2$, and the result follows directly. By Claim 2.1.2 we have that every parallel triple of edges contains at least one odd parallel pair, as the two vertices incident to the parallel triple induce a balanced subgraph with only two edges coming out of it. By Claim 2.1.3 we have that every parallel pair is odd, as the two vertices incident to the parallel triple induce a balanced subgraph with only four edges coming out of it. Hence if $|V(R)| \geq 5$ we have at least three edge-disjoint odd parallel pairs, hence three-edge disjoint odd circuits, which by parity gives four edge-disjoint odd circuits, as desired, as $4 \geq \alpha$. Otherwise $|V(R)| \leq 4$, and the reader can easily verify that if (R, Σ) has at least α many odd edges in every signature then (R, Σ) also has α edge-disjoint odd circuits, as desired.

Case 2: *There is a vertex v not incident with a parallel pair.* Now there are three different ways to split off at v in R. Let (R_1, Σ_1'), (R_2, Σ_2'), and (R_3, Σ_3') be the three possible signed graphs resulting from splitting off at v. We may assume that Σ_1', Σ_2', and Σ_3' are signatures with minimum cardinality. By minimality we have that $|\Sigma_i'| < \alpha$ for $i \in \{1, 2, 3\}$. By Lemma 2.0.2 we have that $|\Sigma_i'| = \alpha - 2$ for $i \in \{1, 2, 3\}$. Now lifting a signature Σ_i through a split adds at most two edges to the signature, so each Σ_i' lifts to a signature Σ_i with $|\Sigma_i| = \alpha$ equivalent to Σ for R, as (R, Σ) has at least α many odd edges in every signature. Furthermore, each Σ_i intersects $\delta(v)$ in exactly two edges, with $\Sigma_i \cap \delta(v)$ not equal to $\Sigma_j \cap \delta(v)$ for $i, j \in \{1, 2, 3\}$. Let $\delta(v) = \{e_1, e_2, e_3, e_4\}$. By resigning by $\delta(v)$ and renaming edges we may assume that $e_4 \in \Sigma_i$ for every $i \in \{1, 2, 3\}$ and that $\Sigma_i \cap \Sigma_j \cap \delta(v) = \{e_4\}$ for all $i, j \in \{1, 2, 3\}$. Hence we have that $e_i \in \Sigma_i$ for every $i \in \{1, 2, 3\}$.

The following two technical claims will be useful. We will first establish a result on how the signatures Σ_i' can intersect.

Claim 2.1.5. *All of the Σ_i' are disjoint, and $|\Sigma_i'| \geq 2$.*

Proof. Suppose not for a contradiction. Hence either there is a pair of non-disjoint sets Σ_i' and Σ_j' or a set Σ_i' with at most one edge. We may assume without loss of generality that if there is a pair of non-disjoint sets that Σ_1' and Σ_2' overlap.

As $|\Sigma_1'| \leq 2$ and $|\Sigma_2'| \leq 2$ as $\alpha \leq 4$ we have that $|\Sigma_1' \Delta \Sigma_2'| \leq 2$. On the other hand, as all the Σ_i' have the same size we have that if one has at most one edge then all have at most one edge, and hence $|\Sigma_1' \Delta \Sigma_2'| \leq 2$.

At any rate, we have that $|\Sigma_1' \Delta \Sigma_2'| \leq 2$. As v is not a cut-vertex by Claim 2.1.4 we have that $\Sigma_1' \neq \Sigma_2'$. As $e_4 \in \Sigma_i$, for $i \in \{1, 2, 3\}$, we have that $|\Sigma_1 \Delta \Sigma_2| \leq 4$. Let $S = \Sigma_1 \Delta \Sigma_2$; this is a cut of R. Now (R, Σ) is equivalent to (R, Σ_1) and S gives a cut of (R, Σ_1) with at most four edges partitioning $V(R)$ into two sides S_1 and S_2, with $v \in S_2$. As v is not in a parallel pair, both sides of this cut have at least two vertices.

Symmetrically, $S \Delta \delta(v)$ is another cut of (R, Σ_1) with at most four edges partitioning $V(R)$ into two sides S_3 and S_4, with $v \in S_4$, where both sides of this cut have at least two vertices. By renaming if necessary we may assume that $S_1 \subseteq S_4$ and $S_3 \subseteq S_2$. As two edges of Σ_1 are in $\delta(v)$ and the third is in $\Sigma_1' \Delta \Sigma_2'$, we have the fourth edge is in exactly one of $R[S_1]$ or $R[S_3]$.

This contradicts Claim 2.1.3, as one of S or $S \Delta \delta(v)$ would be a cut with at most four edges, where both sides have at least two vertices and with one side balanced, as desired. $\square$

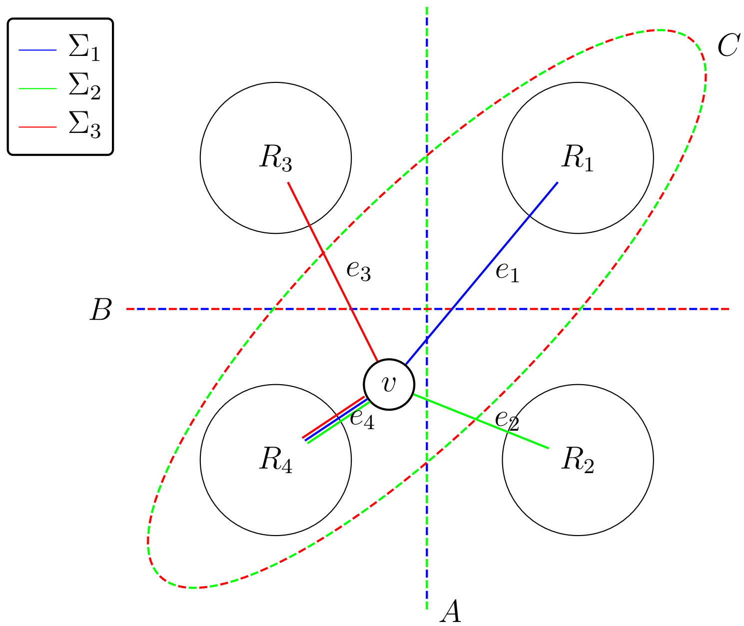

Figure 2.3: Three Cuts of R.

Note that as $\alpha \leq 4$ and $|\Sigma_i'| = 2$, we have that $\alpha = |\Sigma_i| = 4$. Now as $\Sigma_i \Delta \Sigma_j$ are cuts of R, for $i, j \in \{1, 2, 3\}$, we have that $\Sigma_i' \Delta \Sigma_j'$ are four-edge cuts of $R \setminus v$. Let $A' = \Sigma_1' \Delta \Sigma_2'$, $B' = \Sigma_1' \Delta \Sigma_3'$, and $C' = \Sigma_2' \Delta \Sigma_3'$, and let $A = \Sigma_1 \Delta \Sigma_2$, $B = \Sigma_1 \Delta \Sigma_3$, and $C = \Sigma_2 \Delta \Sigma_3$. Now $A' \subseteq A$, $B' \subseteq B$, and $C' \subseteq C$. Furthermore, A, B, and C split R into four non-empty components R_1, R_2, R_3, and R_4, with $|V(R_i)| \geq 1$ and v, connected to R_i via e_i, as shown in Figure 2.3. Note that:

$$\begin{aligned}
&\delta(R_1, R_2) \subseteq B \cap C \subseteq \Sigma_3, \\
&\delta(R_1, R_3) \subseteq A \cap C \subseteq \Sigma_2, \\
&\delta(R_1, R_4) \subseteq A \cap B \subseteq \Sigma_1, \\
&\delta(R_2, R_3) \subseteq A \cap B \subseteq \Sigma_1, \\
&\delta(R_2, R_4) \subseteq A \cap C \subseteq \Sigma_2, \text{ and} \\
&\delta(R_3, R_4) \subseteq B \cap C \subseteq \Sigma_3.
\end{aligned}$$

Now some more technical claims.

Claim 2.1.6. *There is at most one edge between any two of the components R_1, R_2, R_3, and R_4.*

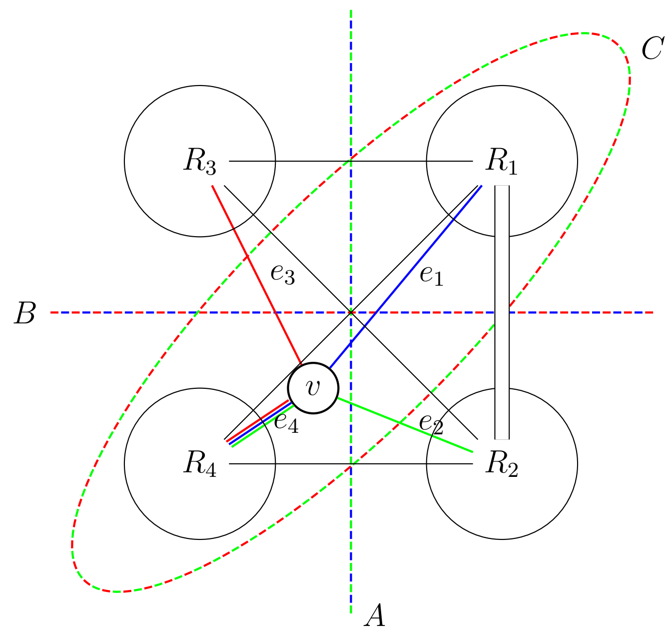

Figure 2.4: Parallel Edges between R_1 and R_2 – Odd Cuts

Proof. Suppose not for a contradiction; we may assume without loss of generality that there are at least two edges between R_1 and R_2. Note that the edges linking R_1 and R_2 are in Σ_3, and that any edge linking R_3 and R_4 is also in Σ_3. Hence there are no edges linking R_3 and R_4, as $|\Sigma_3| = 4$.

As R is Eulerian $|\delta(R_3)|$ is even. Hence there exists i and j in $\{1, 2, 3, 4\}$ such that either Σ'_1 or Σ'_2 consists of two edges, both with one end in R_i and other in R_j. Otherwise, $|\delta(R_3)|$ would be odd, as $\Sigma'_1 \cup \Sigma'_2$ would consist of four edges, one linking R_3 and R_1, one linking R_4 and R_2, one linking R_3 and R_2, and one linking R_1 and R_4, as illustrated in Figure 2.4.

Hence one of $\delta(R_3)$ or $\delta(R_4)$ has at most two edges, a contradiction by Claim 2.1.2, as $|V(R_3)| \geq 1$ and $|V(R_4)| \geq 1$. $\square$

As there are exactly six edges in $\Sigma'_1 \cup \Sigma'_2 \cup \Sigma'_3$, there is exactly one edge linking any two of R_1, R_2, R_3, and R_4. Hence we are in the configuration illustrated in Figure 2.5, where there is a complete graph linking v, R_1, R_2, R_3, and R_4, and where each Σ'_i induces a perfect matching between R_1, R_2, R_3, and R_4, and where each Σ'_i induces a triangle linking v, R_i, and R_4. Since $\Sigma_3 \cap E(R_1) = \Sigma_3 \cap E(R_2) = \Sigma_3 \cap E(R_3) = \Sigma_3 \cap E(R_4) = \varnothing$,

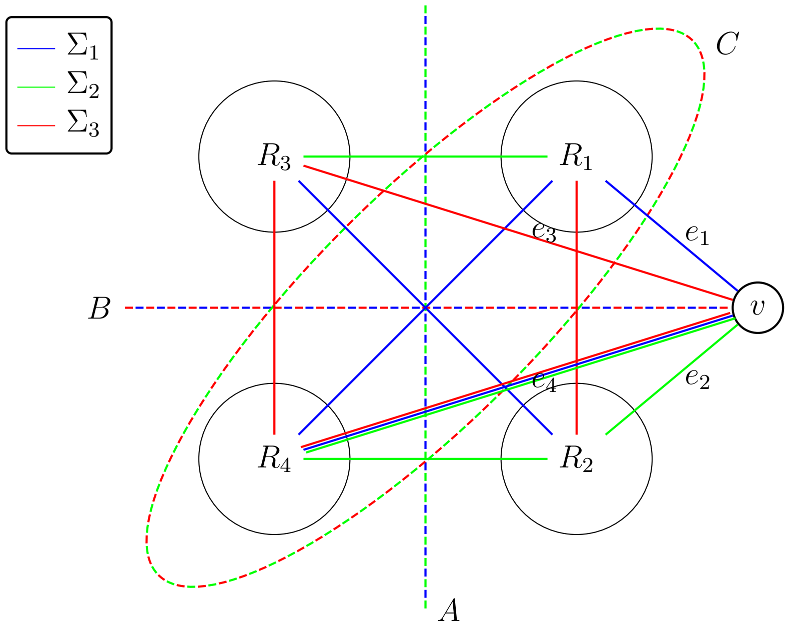

Figure 2.5: Odd K_5.

by Claim 2.1.3 we have that R_1, R_2, R_3 and R_4 contain only a single vertex. Now resigning Σ_3 by $\delta(R_1)$ and $\delta(R_2)$ produces odd-K_5, a contradiction, as desired. $\square$

As a result we obtain the following theorem which characterizes when a signed tour graph can be resigned down to two odd edges.

Lemma 2.1.7. *Let (R, Σ) be a four-regular signed graph with $|\Sigma|$ even. Then either:*

- (R, Σ) *can be resigned to two odd edges, or*
- (R, Σ) *contains odd-K_5 as an immersion minor, or*
- (R, Σ) *contains four edge-disjoint odd circuits.*

2.2 Immersion-Minor-Minimal Graphs

In light of Lemma 2.1.7 we will now characterize what are the immersion-minor-minimal internally-six-edge connected signed graphs with even signature that pack four-edge-disjoint odd circuits.

Before we do so we will state the following results from Chapter 4. A four-regular graph is *weakly six-edge connected* if if it is four-edge connected and every cut on four edges partitions the graph into two components, one of which has at most two vertices.

Lemma 2.2.1. *Let v be a vertex in an internally six-edge connected four-regular graph R. Then two out of the three ways to split off v in R result in weakly six-edge connected graphs.*

Lemma 2.2.2. *Let v be a vertex in an internally six-edge connected four-regular graph R. Either there is a way to split off v in R while remaining internally six-edge connected or v is incident to an edge in three triangles of R.*

Moreover, if R is not isomorphic to K_5, and $\triangle_1$, $\triangle_2$, and $\triangle_3$ are three triangles of R that share an edge, then for all distinct i and j in $\{1, 2, 3\}$, there are two ways to split off at the single vertex in $V(\triangle_i) \setminus V(\triangle_j)$ while remaining internally six-edge connected.

We will also need the following easy observation about weakly six-edge connected tour graphs.

Lemma 2.2.3. *Let R be a weakly six-edge connected tour graph. If R has no parallel pairs, then R is internally six-edge connected.*

Proof. Suppose R was weakly six-edge connected. Then there is a bipartition of $V(R)$ into (A, B), where $|A| = 2$, $|B| \geq 2$, and $|\delta(A)| = 2$. Now $R[A]$ is a connected two-regular graph on two vertices, that is, a parallel pair, a contradiction, as desired. □

We will prove the following lemma.

Lemma 2.2.4. *Let (R, Σ) be an immersion-minor-minimal internally-six-edge signed tour graph packing four-edge-disjoint odd circuits. Then $|V(R)| \leq 7$.*

For brevity, we will say a signed tour graph (R, Σ) is *packed* if it is an immersion-minor-minimal internally-six-edge connected signed tour graph with a packing $\mathcal{P}$ of four edge-disjoint odd circuits. We start by proving the following structural results on packed signed graphs (R, Σ):

Lemma 2.2.5. *Let $\mathcal{P}$ be any packing of four edge-disjoint odd circuits of a packed signed tour graph (R, Σ). Then every vertex in $V(R)$ is covered by two circuits in $\mathcal{P}$, and hence every edge of R is covered by some circuit in $\mathcal{P}$.*

Proof. Suppose not for a contradiction. Let v be a vertex not covered by two odd circuits. Then v has two neighbours w and x also not covered by two odd circuits. Note that as R is four-regular, if v is in an edge incident to three circuits then one of w or x is not. Hence by Lemma 2.2.2 there is a way to split off and remain internally six-edge-connected at one of v, w or x. Such a split preserves the number of odd circuits, contradicting minimality, as desired. ☐

Lemma 2.2.6. *Let (R, Σ) be a packed signed tour graph with packing $\mathcal{P}$. Let $\Delta = \{e_1, e_2, e_3\}$ be an odd triangle in $\mathcal{P}$ and let C be an odd circuit in $\mathcal{P}$, such that C meets Δ in all three vertices. If $C = P_1P_2P_3$ such that P_i is a path and $C_i = P_ie_i$ is a circuit of R, then each C_i is an even circuit.*

Proof. Suppose not for a contradiction. Let v_1, v_2, v_3 be the vertices of Δ, with $e_1 = \{v_1, v_2\}$, $e_2 = \{v_2, v_3\}$, and $e_3 = \{v_3, v_1\}$. First observe that the vertices on the triangle are not incident to an edge in three triangles, so every vertex on the triangle can be split off in a way preserving internal six-connectivity. By resigning on a vertex in the triangle we may assume without loss of generality that e_1 is the only odd edge in Δ. Now as C is odd, it contains an odd path.

Suppose for a contradiction that P_1 is odd. By parity either both P_2 and P_3 are odd or both are even. If both are odd then note that P_2e_2 and P_3e_3 are odd. Now both possible ways are splitting off at v_2 that do not introduce a parallel pair preserve the number of odd circuits, as both P_2e_2, P_3e_3 are odd and both $P_2P_3e_1$, $P_1e_2e_3$ are odd. The split at v which preserves internal six-connectivity is one of the two splits that does not introduce a parallel pair, hence there is a way to split off at v while keeping four edge-disjoint odd circuits and internal six-connectivity, a contradiction.

Similarly, if both P_2 and P_3 are even both ways of splitting off at v_2 that do not introduce a parallel pair keep four edge-disjoint odd circuits, as both $P_1e_2e_3$, $e_1P_2P_3$ are odd and so are P_3e_3 and P_2e_2, a contradiction. Hence P_1 cannot be odd and so exactly one of P_2, P_3 is; by symmetry we may assume P_2 is odd.

Now both ways of splitting off at v_3 that do not introduce a parallel pair preserve the number of odd circuits, as both P_1e_1, P_3e_3 are odd and both $P_2e_3P_1$, $e_1e_2P_3$ are odd. Hence there is a way to split off at v_3 while keeping four edge-disjoint odd circuits and internal six-edge connectivity, contradicting minimality, as desired. ☐

Lemma 2.2.7. *If C_1 and C_2 are two edge-disjoint odd circuits of a signed tour graph (R, Σ) that meet at two vertices v and w, then either there are two distinct edge-disjoint odd circuits C_1' and C_2' using edges of C_1 and C_2 that use a transition different from the one C_1 and C_2 use at v.*

Proof. Suppose C_1 and C_2 meet twice. Then $C_1 = vP_1wP_2$ for paths P_1 and P_2 and $C_2 = vP_3wP_4$ for paths P_3 and P_4. Now as C_1 is odd, one of P_1 or P_2 is odd; we may assume without loss of generality that P_1 is odd. Similarly, we may assume that P_3 is odd. Now P_1P_4 is an odd closed walk which is edge-disjoint from the odd closed walk that is P_2P_3. Note that these odd walks use different transitions at v. As P_1P_4 is an odd closed walk, it contains an odd cycle C_1'; similarly P_2P_3 contains an odd cycle C_2'. Note that if v is in both C_1' and C_2' they use a different transition; otherwise, we have two odd cycles C_1' and C_2' using the edges of C_1 and C_2 that do not fully cover v.

However, if v is not in C_1' or v is not in C_2', $\mathcal{P}\Delta\{C_1, C_2, C_1', C_2'\}$ gives a packing of four edge-disjoint odd circuits that does not cover every edge incident to v, a contradiction. Hence $v \in V(C_1')$ and $v \in V(C_2')$, as desired. $\square$

We will need a few more technical propositions:

Lemma 2.2.8. *Let $\mathcal{P}$ be any packing of four edge-disjoint odd circuits of a packed signed tour graph (R, Σ). If $\triangle$ is an odd triangle of $\mathcal{P}$, then there is no edge $e \in \triangle$ that is contained in two other odd triangles $\triangle_1, \triangle_2$ of (R, Σ).*

Proof. Suppose not for a contradiction. As $\triangle_2$ and $\triangle_3$ are both odd circuits,

$$\begin{aligned} |\triangle_2 \cap \Sigma| &= |(\triangle_2 \setminus \{e\}) \cap \Sigma| + |\{e\} \cap \Sigma| \equiv 1 \pmod 2, \text{ and} \\ |\triangle_3 \cap \Sigma| &= |(\triangle_3 \setminus \{e\}) \cap \Sigma| + |\{e\} \cap \Sigma| \equiv 1 \pmod 2. \end{aligned}$$

Consider the circuit $C = \triangle_2 \Delta \triangle_3$. This circuit is even, since

$$|C \cap \Sigma| = |(\triangle_2 \setminus \{e\}) \cap \Sigma| + |(\triangle_3 \setminus \{e\}) \cap \Sigma| \equiv 0 \pmod 2.$$

Hence $C \notin \mathcal{P}$. Let w be the vertex in $\triangle_2$ that is not in either $\triangle$ or $\triangle_3$, and let Q and S be the two odd circuits in $\mathcal{P}$ that are incident to w. As R is not isomorphic to K_5, by Lemma 2.2.2, w can be split apart in two ways preserving internal six-edge connectivity. These two ways to split are the two ways to split at w that do not identify $\triangle_2$ to a a parallel pair. However, since $S \neq C$ and $Q \neq C$, S and Q are immersed to edge-disjoint odd circuits in one of the two resulting graphs. Hence (R, Σ) is not minimal, which is a contradiction, as (R, Σ) is a packed signed tour graph.

$\square$

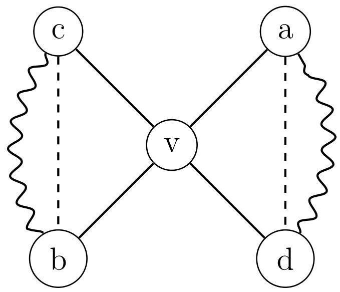

Figure 2.6: Circuits C and D.

Lemma 2.2.9. *Let (R, Σ) be a packed signed tour graph with packing $\mathcal{P}$. If C and D are two odd circuits of $\mathcal{P}$ that share at least two common vertices, then either C is a triangle or D is a triangle.*

Proof. Suppose not for a contradiction, so $|C| \geq 4$ and $|D| \geq 4$. By Lemma 2.2.7 there is an alternate transition at v giving rise to two other edge-disjoint odd circuits D' and C' using $E(C \cup D)$.

By Lemma 2.2.1 there are two ways to split off at v preserving weak six-connectivity. Hence there is one way to split off at v which preserves weak six-connectivity and also preserves the two odd circuits C and D (possibly by replacing C and D with C' and D' beforehand). Let (R', Σ') be the graph obtained by splitting off at v.

Consider the neighbours of v in R; let the neighbours of v in C be c and b, and let the neighbours of v in D be a and d. As a split which preserves weak but not internal six-connectivity in an internally six-edge-connected graph is one which turns a triangle into a parallel pair, we may assume by symmetry at least one of $\triangle_1 = \{c, b, v\}$ or $\triangle_2 = \{a, d, v\}$ is a triangle in R, as shown in Figure 2.6. Otherwise (R', Σ') would be internally six-edge-connected, a contradiction, as desired. Furthermore, as R' is weakly six-edge-connected, we have that if $\triangle_1$ is a triangle in R, then the $\{c, b\}$ edge is not in any triangle of R' and symmetrically with $\triangle_2$ and the $\{a, d\}$ edge.

As $C \neq \triangle_1$ and $D \neq \triangle_2$, from $\mathcal{P}$ we obtain a packing $\mathcal{P}'$ of four edge-disjoint odd circuits of (R', Σ') such that neither the parallel pair $\triangle_1' = \{c, b\}$ nor the parallel pair $\triangle_2' = \{a, d\}$ are in $\mathcal{P}'$. Hence we may split off again at possibly b or d to eliminate the parallel pairs in one of the two ways that do not introduce loops to obtain a third signed graph (R'', Σ''). As neither of the parallel pairs were in $\mathcal{P}'$, this does not disturb the parity of the cycles in $\mathcal{P}'$, so $\mathcal{P}'$ gives a packing of four edge-disjoint odd cycles of (R'', Σ'').

Note that this splitting off operation at b or d does not disturb weak six-edge-connectivity; all we have done is shrunk a parallel pair to a single vertex.

As splitting off at v can only introduce up to two parallel pairs, namely $\{c, b\}$ or $\{a, d\}$, (R'', Σ'') is a weakly six-edge connected graph with no parallel pairs. Hence (R'', Σ'') is internally six-edge connected, and thus it is a packed signed tour graph, a contradiction, as desired. $\square$

We are now ready to prove Lemma 2.2.4.

Proof of Lemma 2.2.4. Let (R, Σ) be a packed signed tour graph.

Claim 2.2.10. $|V(R)| \leq 9$, *and* $\mathcal{P}$ *consists of three triangles and one cycle.*

Proof. Suppose that $|V(R)| \geq 10$ for a contradiction. Let $\mathcal{P} = \{C_1, C_2, C_3, C_4\}$. Every vertex in R is incident to a pair of cycles. As there are at least 10 vertices, and only $\binom{4}{2} = 6$ possible such pairs, there are at least two vertices who are incident to the same pair of cycles. Without loss of generality we may assume that C_1 and C_2 meet twice.

Hence by Lemma 2.2.9 either C_1 or C_2 is a triangle; we may assume that C_1 is without loss of generality. Each of the remaining vertices not in C_1 are incident to a pair of cycles in $\{C_2, C_3, C_4\}$. As there are at least 7 remaining vertices and only $\binom{3}{2} = 3$ possible pairs, we have that two of C_2, C_3, and C_4 meet twice. Again one is a triangle; we may assume that C_2 is without loss of generality.

Finally, each of the remaining vertices not in C_1 nor in C_2 are incident to a pair of cycles in $\{C_3, C_4\}$. As there are at least three vertices remaining and $\binom{2}{2} = 1$ possible pair, C_3 and C_4 meet twice; hence one is a triangle. Hence $\mathcal{P}$ consists of three triangles and one other cycle. Now as every vertex is covered by two cycles, there can be at most $3 \times 3 = 9$ vertices in R, as desired. $\square$

Hence we have shown that $|V(R)| \leq 9$. So (R, Σ) consists of four edge-disjoint odd circuits, three of which are edge-disjoint triangles and the fourth which meets every triangle. Let $\triangle_1, \triangle_2$, and $\triangle_3$ be the three odd triangles in $\mathcal{P}$, and let C be the remaining odd circuit. Note that $|V(R)| \geq 6$ as R is internally six-edge-connected and hence simple. It remains to show that $|V(R)| \leq 7$. To do so, we will prove the following three claims.

Claim 2.2.11. *Let* $\triangle'$ *be a triangle of* $\mathcal{P}$. *If* C *meets* $\triangle'$ *three times, then there is no triangle* $\triangle$ *of* R *such that* $\triangle$ *consists of two edges of* C *and one edge of* $\triangle'$.

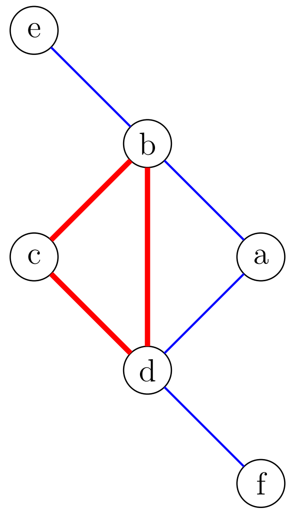

Figure 2.7: $\triangle' = \{b, c, d\}$, $\triangle = \{a, b, d\}$, and $C = \ldots e, b, a, d, f \ldots$

Proof. Suppose not for a contradiction. Hence we are the configuration shown in Figure 2.7. As R is simple, any two edge-disjoint odd triangles may only share a single vertex. Hence $|V(C)| > 4$. Now d cannot be incident to an edge in three triangles of R, as this third triangle would use the edges of C, since any two triangles in $\mathcal{P}$ meet in at most a single vertex. This would imply that $e = f$ and hence $|V(C)| = 4$, contradicting the fact that $|V(C)| > 4$.

Hence there is a way to split off at d to remain internally six-edge-connected. As a four-regular graph that is internally six-edge-connected must be simple, such a split must identify the edges $(b, d), (d, f)$ and $(c, d), (a, d)$. By Lemma 2.2.6 we have that $\triangle$ is even, and hence this split preserves the number of edge disjoint odd circuits, a contradiction, as desired. $\square$

Claim 2.2.12. *$|V(R)| \neq 9$.*

Proof. Suppose not for a contradiction. Hence R is a four-regular graph on nine-vertices that has an Hamiltonian cycle C and three edge-disjoint odd triangles. From Claim 2.2.11, as C meets every odd triangle three times, there is no other triangle in R. Hence R is the Cayley graph $X(\mathbb{Z}_9, \{\pm 1, \pm 3\})$, shown in Figure 2.8. By renumbering we may assume that $C = (1, 2, 3, \ldots, 9)$.

Claim 2.2.13. *Every vertex in $X(\mathbb{Z}_9, \{\pm 1, \pm 3\})$ can be split off in two ways preserving internal six-edge-connectivity.*

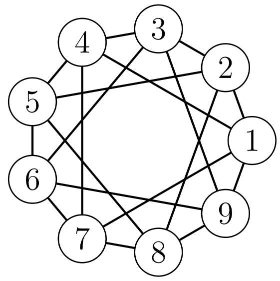

Figure 2.8: $X(\mathbb{Z}_9, \{\pm 1, \pm 3\})$

Proof. There are three ways to split off any vertex in $X(\mathbb{Z}_9, \{\pm 1, \pm 3\})$. One split results in a graph with a graph with a parallel pair, and the other two give simple graphs on eight vertices that do not consist of two copies of K_4 joined by four edges. Hence by Lemma 4.4.8 there are two ways to split off preserving internal six-connectivity. □

Hence there are two ways to split off at every vertex preserving internal six-connectivity. Since C meets each triangle three times, by Lemma 2.2.7 there is a way of splitting at a vertex preserving both internal six-connectivity and the number of odd circuits, a contradiction, as desired. □

Claim 2.2.14. *$|V(R)| \neq 8$.*

Proof. Suppose not for a contradiction. Now exactly two of the triangles in $\mathcal{P}$ share a vertex as every vertex in R is covered by two odd circuits of $\mathcal{P}$; we may assume without loss of generality that $\triangle_1$ and $\triangle_2$ share a vertex, say d. Let $\triangle_3 = \{a, b, c\}$, $\triangle_2 = \{d, f, g\}$, and $\triangle_1 = \{d, e, h\}$.

As R is internally six-edge connected, there at least six edges from $\triangle_3$ to the rest of the graph. There are at most six edges from $\triangle_3$ to the rest of the graph as two of the edges incident to each of a, b, and c are in the triangle. Hence either $\{e, f\}$ is an edge or $\{g, h\}$ is an edge of R. This edge is also an edge of C as it is not in any of the triangles. We may assume without loss of generality that $\{g, h\}$ is an edge of C by symmetry. Now g is incident to one more edge of C. This edge cannot be from g to e as otherwise $|\delta(\{d, e, f, g, h\})| < 6$, a contradiction. Hence it is from g to one of a, b, or c. By symmetry we may assume that C contains a $\{g, b\}$ edge.

Now as R is simple and $|V(C)| \geq 3$, there is no $\{b, h\}$ edge. Two cases follow.

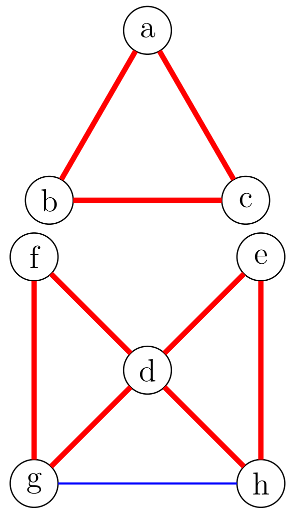

Figure 2.9: Counterexample on Eight Vertices. Odd triangles drawn in thick red, C drawn in blue.

Case 1: *There is an edge of C linking b to f.* Now there cannot be an edge of C linking f to e, as otherwise $|\delta(\{d, e, f, g, h\})| = 4$, contradicting internal six-connectivity. Hence there is an edge of C linking f to a or c, and by symmetry we may assume there is an edge of C linking f to a. As $|C| = 8$, there is no edge of C linking a to h; hence the remaining edges of C must be from a to e, from e to c, and from c to h, as shown in Figure 2.10. However the triangle $\{a, e, c\}$ contradicts Claim 2.2.11, a contradiction, as desired.

Case 2: *There is an edge of C linking b to e.* Now there cannot be an edge of C linking f to e, as otherwise $|\delta(\{d, e, f, g, h\})| = 4$, contradicting internal six-connectivity. Hence there is an edge of C linking e to a or c, by symmetry we may assume there is an edge of C linking e to a. As $|C| = 8$, there is no edge of C linking a to h; hence the remaining edges of C must be from a to f, from f to c, and from c to h, as shown in Figure 2.11. However the triangle $\{a, f, c\}$ contradicts Claim 2.2.11, a contradiction, as desired. $\square$

Hence $|V(R)| \leq 7$, as desired. $\square$

Now we will show that R is either $R(F_7)$ or $R(W_7)$.

Lemma 2.2.15. *If $|V(R)| = 6$ then (R, Σ) is equivalent to $R(R_7)$.*

Proof. The unique simple four-regular graph on six vertices is the octrahedal graph, shown in Figure 2.12. Up to symmetry there is exactly one set of four edge-disjoint odd triangles

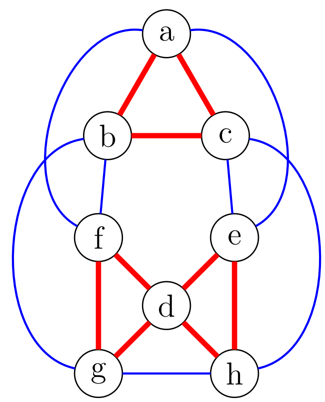

Figure 2.10: Counterexample on Eight Vertices - Case 1.

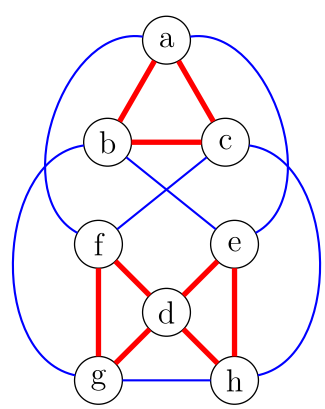

Figure 2.11: Counterexample on Eight Vertices - Case 2.

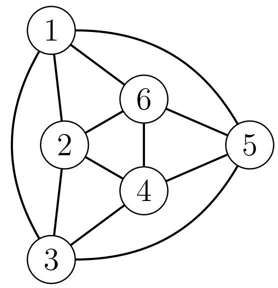

Figure 2.12: Octahedral Graph.

- the triangles $\{1,3,5\}$, $\{1,2,6\}$, $\{2,3,4\}$, and $\{4,5,6\}$. By parity as (R,Σ) has an even number odd edges it follows that exactly zero, two, or four out of the four remaining triangles are odd. We take these three cases in turn.

Case 1: *Zero of the four remaining triangles are odd.* We may resign (R,Σ) so that the spanning tree given by $(1,2)$, $(2,3)$, $(3,4)$, $(4,5)$, and $(5,6)$ is even. Hence $(2,4)$ is odd, $(4,6)$ is odd, $(2,6)$ is even, $(1,6)$ is odd, $(1,5)$ is odd, $(3,5)$ is even, and $(1,3)$ is even, giving us $R(F_7)$, as desired.

Case 2: *Two of the four remaining triangles are odd.* By symmetry we may assume without loss of generality that the triangles $\{1,2,3\}$ and $\{1,5,6\}$ are odd. We may resign (R,Σ) so that the spanning tree given by $(1,2)$, $(1,3)$, $(3,4)$, $(4,5)$, and $(4,6)$ is even. Hence the edge $(2,6)$ is odd, $(2,4)$ is odd, $(6,5)$ is odd, $(1,2)$ odd, and all other edges are even. However this graph admits an odd-K_5-immersion minor, a contradiction, as desired.

Case 3: *All of the remaining triangles are odd.* We may resign (R,Σ) so that the spanning tree given by $(1,2)$, $(2,3)$, $(3,4)$, $(4,5)$, and $(5,6)$ is even. However this graph admits an odd-K_5-immersion minor, a contradiction, as desired. □

Lemma 2.2.16. *If $|V(R)| = 7$ then (R,Σ) is equivalent to $R(W_7)$.*

Proof. Note that there are exactly two unique simple four-regular graphs on seven vertices, as shown in Figures 2.13 and 2.14.

Claim 2.2.17. *R is not the Postman's Work Day Graph, as shown in Figure 2.14.*

Proof. Suppose not for a contradiction. Let $\triangle_1, \triangle_2$ be two edge-disjoint odd triangles in $\mathcal{P}$, and let C_3, C_4 be the other two odd four-cycles in $\mathcal{P}$. There are two cases:

Case 1: $\triangle_1$ *and* $\triangle_2$ *share a vertex.* By symmetry we may assume without loss of generality that $\triangle_1 = \{c,b,g\}$ and $\triangle_2 = \{e,f,g\}$. Hence C_3 and C_4 meet twice, once at a and once at

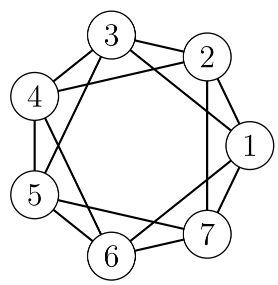

Figure 2.13: Circulant $X(\mathbb{Z}_7, \{\pm 1, \pm 2\})$.

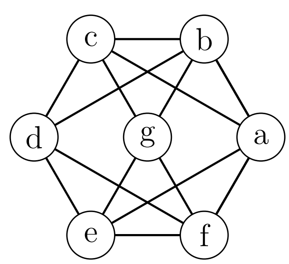

Figure 2.14: Postman's Work Day Graph.

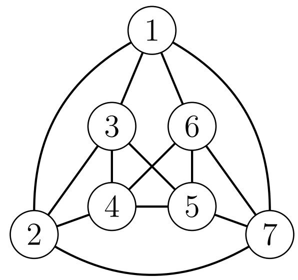

Figure 2.15: An alternate drawing of $X(\mathbb{Z}_7, \{\pm 1, \pm 2\})$.

d. Note that there are two ways to split off at a preserving internal six-connectivity. Hence by Lemma 2.2.7 there is way to split off at a preserving both internal six-connectivity and the number of edge-disjoint odd circuits, a contradiction, as desired.

Case 2: *$\triangle_1$ and $\triangle_2$ do not share a vertex.* By symmetry we may assume without loss of generality that $\triangle_1 = \{a, b, c\}$ and $\triangle_2 = \{d, e, f\}$, and that C_3 shares two vertices with $\triangle_1$ and C_4 shares two vertices with $\triangle_2$. By resigning along a spanning tree we may assume that the edges (d, e), (d, f), (c, d), (b, d), (a, e), and (g, e) are all even. Hence the edges (e, f) and (b, c) are odd. We consider two subcases.

Case 2.2.1: *One of $\{c, d, b\}$ or $\{a, e, f\}$ is even.* By symmetry we may assume that $\triangle_5 = \{c, d, b\}$ is even. Hence by parity we have that $C_6 = \{a, b, c, g\}$ is an odd circuit. Now as $\triangle_2 = \{d, e, f\}$ is odd, the closed walk $W = \triangle_2 \cup \triangle_5$ is odd. Hence we may split off at d to get a graph (T', Σ') that is both internally six-edge-connected and contains at least three edge-disjoint odd circuits. By parity (T', Σ') has at least four edge-disjoint odd circuits, contradicting the minimality of G, as desired.

Case 2.2.2: *Both $\{c, d, b\}$ and $\{a, e, f\}$ are odd.* Since three odd triangles cannot share an edge, we have that both $\triangle_5 = \{c, b, g\}$ and $\triangle_6 = \{e, f, g\}$ are even. Hence $\triangle_2' = \{a, e, f\}$ is odd and $C_4' = \{d, e, g, f\}$ is odd. Thus $\mathcal{P}' = \{\triangle_1, \triangle_2', C_3, C_4'\}$ is a set of four-edge disjoint odd circuits of (T, Σ) where the triangles share a vertex, and therefore we obtain our desired contradiction via a reduction by **Case 1**, as desired. $\square$

Hence R is the circulant $X(\mathbb{Z}_7, \{\pm 1, \pm 2\})$. Observe that $\mathcal{P}$ cannot contain two odd triangles and two odd four-cycles, as deleting any pair of edge-disjoint triangles leaves a graph with a five-cycle and a triangle. Taken all of the triangles modulo 7, all of the triangles in R are of the form $\{x, x+1, x+2\}$. Let $\triangle_x$ denote the triangle $\{x, x+1, x+2\}$.

Claim 2.2.18. *Every triangle in (R, Σ) is odd.*

Proof. Suppose not for a contradiction. We say two triangles $\triangle_x, \triangle_y$ are adjacent if $x \pm 2 \equiv y \pmod 7$. Now every packing of four edge-disjoint odd circuits of R consists of three consecutive adjacent triangles and an odd five-cycle. Hence we can find a packing $\mathcal{P}$ of four edge-disjoint circuits of R where one of the triangles in $\mathcal{P}$ is adjacent to an even triangle. Since R is a circulant, by symmetry we may assume without loss of generality that the three triangles in R are $\triangle_2 = \{2,3,4\}$, $\triangle_7 = \{1,2,7\}$, and $\triangle_5 = \{5,6,7\}$. Hence one of the triangles $\{3,4,5\}$ or $\{4,5,6\}$ is even. By symmetry we may assume that $\{3,4,5\}$ is even. Then the parity on the edge $(3,4)$ is the same as the parity of the path $(3,5,4)$. Thus $\triangle_2' = \{2,3,5,4\}$ is odd and therefore the split at vertex 3 which preserves internal six-edge connectivity also preserves the circuits in $\mathcal{P}$, a contradiction, as desired. □

Now to show that (R,Σ) is resignable to $R(W_7)$, first note that we may resign (R,Σ) in a way such that the spanning tree given by the edges $(1,2)$, $(1,3)$, $(1,6)$, $(1,7)$, $(3,4)$, and $(5,6)$ are even. This forces the edges $(3,2)$, $(2,7)$, and $(6,7)$ to be odd. Furthermore, as $|\Sigma|$ is even, the edges $(3,5)$ and $(4,6)$ are even and the edge $(4,5)$ is odd. Now by resigning along the cut given by $\{(1,2),(1,3),(1,6),(1,7)\}$ and by the cut $\{(2,4),(3,4),(4,6),(3,5),(5,6),(5,7)\}$ we obtain a signed graph based on R where every edge is odd, which is exactly $R(W_7)$, as desired. □

Hence we obtain the following lemma:

Lemma 2.2.19. *If (R,Σ) is signed, immersion-minor-minimal internally six-edge connected tour graph packing four edge-disjoint odd circuits with $|\Sigma|$ even then:*

- *$|V(R)| = 6$ and (R,Σ) is equivalent to $R(W_7)$, as shown in Figure 1.16.*
- *$|V(R)| = 7$ and (R,Σ) is equivalent to $R(F_7)$, as shown in Figure 1.17.*

Chapter 3

Preliminaries

In the rest of this thesis we build up the necessary machinery in order to prove Bouchet's Theorem from Lemmas 2.2.19 and 2.1.7. In this chapter we review preliminary results on vertex minors and splits. For each result give a reference in which the result and proof first appeared in.

3.1 Vertex Minors

Note that there are three distinct ways to split off a vertex in a four-regular graph. More generally, for vertex minors, there are three ways to remove a vertex up to local equivalence. This result is due to Bouchet who proved it in the context of isotropic systems. We present a purely graph-theoretic proof of this result due to Geelen and Oum in [9].

Lemma 3.1.1 (Bouchet [2])**.** *Let v and w be two adjacent vertices in a simple graph G. If H is a vertex minor of G with $v \notin V(H)$, then H is a vertex minor of one of $G \setminus v$, $G * v \setminus v$, and $G * v * w * v \setminus v$.*

Note that for any two neighbours u, w of v we have that

$$G * v * u * v = G * v * w * v * u * w * u. \tag{3.1}$$

Hence $GG * v * u * v \setminus v$ is locally equivalent to $G * v * w * v \setminus v$. In light of this fact we will write $G \circ v$ for $G * v * u * v \setminus v$; this is well-defined up to local equivalence. If v has no neighbours then we take $G \circ v = G \setminus v$. For notational convenience we will also write G/v for $G * v \setminus v$.

We first defer this proof to prove the following technical claim, again, due to Geelen and Oum in [9].

Lemma 3.1.2 (Geelen and Oum, [9, Lemma 3.1])**.** *Let G be a simple graph, let v and w be two distinct vertices in G.*

1. *If v is not adjacent to w, then $G * w \setminus v$, $G * w/v$, and $G * w \circ v$ are locally equivalent to $G \setminus v$, G/v, and $G \circ v$ respectively.*
2. *If v is adjacent to w, then $G * w \setminus v$, $G * w/v$, and $G * w \circ v$ are locally equivalent to $G \setminus v$, $G \circ v$, and G/v respectively.*

Proof. It is clear that $G * w \setminus v = G \setminus v * w$ and hence that $G * w \setminus v$ is locally equivalent to $G \setminus v$.

Consider the case where w is adjacent to v. First observe that for two neighbours v and w in G that

$$G * v * w * v = G * w * v * w.$$

See Figure 3.1 for an example of a pivot.

Hence we have that:

$$\begin{aligned}
G * w * v \setminus v &= G * w * v * w * w \setminus v \\
&= (G * v * w * v) * w \setminus v \\
&= [(G * v * w * v) \setminus v] * w.
\end{aligned}$$

Also, we have that, for any neighbour u of v:

$$\begin{aligned}
G * w * u * v * u \setminus v &= G * w * w * v * w * u * w * u \setminus v \\
&= G * v * w * u * w * u \setminus v \\
&= [G * v \setminus v] * w * u * w * u.
\end{aligned}$$

So we have that $G * w \setminus v$, $G * w * v \setminus v$, and $G * w * v * u * v \setminus v$ are locally equivalent to $G \setminus v$, $G * v \setminus v$, and $G * v * u * v \setminus v$ respectively for any neighbour u of v.

Now consider the case where w is not adjacent to v. Now we have that:

$$\begin{aligned} G * w * v \setminus v &= G * v * w \setminus v \\ &= G * v \setminus v * w. \end{aligned}$$

Finally, let u be a neighbour of v. If u is not adjacent to w, then:

$$\begin{aligned} G * w * v * u * v \setminus v &= G * v * u * v * w \setminus v \\ &= [G * v * u * v \setminus v] * w. \end{aligned}$$

Now if u is adjacent to w, then $G*w*u*v*u\setminus v$ is locally equivalent to $G*w*u*v\setminus v$. Now:

$$\begin{aligned} G * w * u * v \setminus v * w &= G * w * u * v * w \setminus v \\ &= [(G * w * u * w) * w * v * w] \setminus v \\ &= [(G * u * w * u) * v * w * v] \setminus v \\ &= G * v * u * v \setminus v \text{ from equation 3.1.} \end{aligned}$$

□

Now we will prove Lemma 3.1.1. For notational convenience, given a string of vertex labels $S = s_1 s_2 \dots s_m$, we let $G * S = G * s_1 * s_2 \dots * s_m$

Proof of Lemma 3.1.1. Let H be a vertex minor of G. Then H is an induced subgraph of a graph G' locally equivalent to G. Now $G' = G*S$ for $S \in V(G)^*$. Now let $v \in V(G)\setminus V(H)$. It suffices to show $G' \setminus v$ is locally equivalent to one of $G \setminus v$, $G * v \setminus v$, or $G * u * v * u \setminus v$ for a neighbour u of v.

This we will do by induction on $|S|$. The base case when $|S| = 1$ follows by definition so we may assume that $|S| \geq 2$. Hence let $S = S'xy$. If v is not y then we have that

$$G * S \setminus v = G * S'xy \setminus v = G * S'x \setminus v * y.$$

Now $(G*S'x)\setminus v$ is locally equivalent to one of $G\setminus v$, G/v, or $G\circ v$. Thus since $G*S'x\setminus v*y$ is locally equivalent to $(G * S'x) \setminus v$, by closure $(G * S'x) \setminus v$ is locally equivlanet to one of $G \setminus v$, G/v, or $G \circ v$, as desired.

Hence we may assume that $S = S'xv$. Now let $G'' = G * S'$. By Lemma 3.1.2 we have that $G'' * xv$ is a vertex minor of one of $G'' \setminus v$, $G'' * v \setminus v$, or $G'' * u * v * u \setminus v$ for some neighbour u of v. By induction as $|S'| < |S|$ and $G'' = G * S'$ we have that $G'' \setminus v$, $G'' * v \setminus v$, and $G'' * u * v * u \setminus v$ are each locally equivalent to one of $G \setminus v$, $G * v \setminus v$, or $G * u * v * u \setminus v$ for some neighbour u of v, which concludes the induction, as desired. □

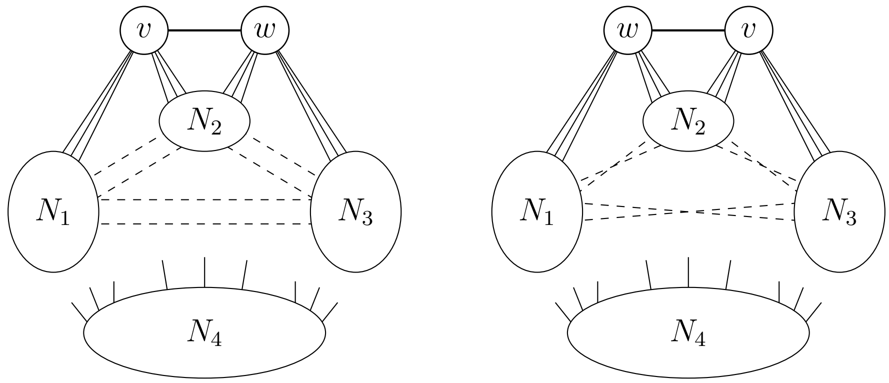

Figure 3.1: Pivoting G at vw; crossed edges are complemented.

3.2 Rank Inequalities

The next result we state and prove is a submodularity inequality on the rank function of a matrix over GF(2). We will use this result in the next section to establish some useful inequalities on the rank of submatrices of the adjacency matrix of a graph G. We will let r denote the matrix rank function. We will also use $M[R]$ to denote the submatrix of M given by the rows in R and $M[R, C]$ to denote the submatrix of M given by the rows in R and the columns in C.

Lemma 3.2.1 (Truemper, [18, Lemma 2.3.11]). *Let M be a $n \times m$ binary matrix indexed by rows R and columns C. Then for any sets X_1, Y_1 of rows and X_2, Y_2 of columns,*

$$r(M[X_1, X_2]) + r(M[Y_1, Y_2]) \geq r(M[X_1 \cup Y_1, X_2 \cap Y_2]) + r(M[X_1 \cap Y_1, X_2 \cup Y_2]).$$

Proof. We may assume without loss of generality that R and C are disjoint sets. Now consider the $n \times (n + m)$ binary matrix $M' = [I_{n \times n}|M]$ with rows indexed by R and columns given by $R \cup C$. We have that:

$$\begin{aligned}
r(M[X_1, X_2]) &= r(M'[R, (R \setminus X_1) \cup X_2]) - |R \setminus X_1| \\
r(M[Y_1, Y_2]) &= r(M'[R, (R \setminus Y_1) \cup Y_2]) - |R \setminus Y_1| \\
r(M[X_1 \cap Y_1, X_2 \cup Y_2]) &= r(M'[R, (R \setminus (X_1 \cap Y_1)) \cup (X_2 \cup Y_2)]) - |R \setminus (X_1 \cap Y_1)| \\
r(M[X_1 \cup Y_1, X_2 \cap Y_2]) &= r(M'[R, (R \setminus (X_1 \cup Y_1)) \cup (X_2 \cap Y_2)]) - |R \setminus (X_1 \cup Y_1)|.
\end{aligned}$$

By submodularity we have that:

$$\begin{aligned}&r(M'[R,(R\setminus X_1)\cup X_2]) + r(M'[R,(R\setminus Y_1)\cup Y_2])\\&\geq r(M'[R,(R\setminus(X_1\cap Y_1))\cup(X_2\cup Y_2)]) + r(M'[R,(R\setminus(X_1\cup Y_1))\cup(X_2\cap Y_2)]).\end{aligned}$$

Moreover,

$$|R\setminus X_1| + |R\setminus Y_1| = |R\setminus(X_1\cap Y_1)| + |R\setminus(X_1\cup Y_1))|.$$

Hence

$$r(M[X_1,X_2]) + r(M[Y_1,Y_2]) \geq r(M[X_1\cup Y_1, X_2\cap Y_2]) + r(M[X_1\cap Y_1, X_2\cup Y_2]),$$

as desired. □

Chapter 4

Generating Prime Graphs

By Lemma 1.4.5 the excluded minors for the class of circle graphs are prime. In this chapter we will prove Lemma 1.4.5, which will give us an inductive tool for studying prime graphs. For clarity we restate that lemma now.

Lemma 4.0.1 (Bouchet [3])**.** *Let G be a prime graph. Either G is locally equivalent to C_5 or there is a graph G' locally equivalent to G such that $G' \setminus v$ is prime for some vertex $v \in V(G)$.*

4.1 Cut Rank

We start by introducing a connectivity function. Let G be a simple graph, and $A(G)$ be its adjacency matrix. The *cut rank function* of a graph G, denoted $\rho_G(X)$, is the rank of the matrix $A(G)[X, V(G) \setminus X]$ taken over GF(2); see Oum [16]. We write $\rho(X)$ when it is clear which graph is being used.

Splits are innately related to cut-rank.

Lemma 4.1.1. *Let G be a simple graph, $A \subseteq V(G)$. Then $(A, V(G) \setminus A)$ is a split of G if and only if $|A| \geq 2$, $|V(G) \setminus A| \geq 2$, and $\rho(A) = 1$.*

Proof. Suppose $(A, V(G) \setminus A)$ is a split of G. Then for every $x \in A$, $N(x) \cap (V(G) \setminus A)$ is either empty of $N(A) \cap (V(G) \setminus A)$. Hence $r(A(G)[A, V(G) \setminus A] = 1$.

Conversely, suppose that $A \subseteq V(G)$ satisifies $|A| \geq 2$, $|V(G) \setminus A| \geq 2$, and $\rho(A) = 1$. Then the rows of $A(G)[A, V(G) \setminus A]$ are co-linear. Since this rank is taken over GF(2), this means that there exists a $w \in \{0,1\}^{V(G)\setminus A}$ such that every row of $A(G)[A, V(G) \setminus A]$ is either $0w$ or $1w$. Hence the neighbourhood of every vertex in A in $V(G) \setminus A$ is either empty or the subset of $V(G) \setminus A$ supporting w, as desired. $\square$

Furthermore, the cut rank function is invariant under local complementation.

Lemma 4.1.2 (Oum [16]). *Let v be a vertex in a simple graph G, and let $G' = G * v$. Then for every $X \subseteq V(G)$, $\rho_G(X) = \rho_{G'}(X)$.*

Proof. As $\rho'_G(X) = \rho'_G(V(G) \setminus X)$, we may assume that $v \in X$.

Now $A(G * v)[X, V(G) \setminus X]$ is obtained from $A(G)[X, V(G) \setminus X]$ by adding the row for v to the rows for every neighbour of v in X. These are elementary row operations, and hence the rank of $A(G*v)[X, V(G) \setminus X]$ is the same as the rank of $A(G)[X, V(G) \setminus X]$. $\square$

We would like to know how splits, and hence cut-rank, behave under taking vertex-minors. The following equalities will prove useful in doing so.

Lemma 4.1.3 (Oum [16]). *Let G be a simple graph, $v \in V(G)$, and $X \subseteq V(G) \setminus \{v\}$. Then*

$$\rho_{G/v}(X) = r\left(\left[\begin{array}{c|c} 1 & A(G)[\{v\}, (V(G) \setminus X) \setminus \{v\}] \\ \hline A(G)[X, \{v\}] & A(G)[X, (V(G) \setminus X) \setminus \{v\}] \end{array}\right]\right) - 1.$$

Proof. Let $V = V(G)$, $N = N_G(v)$, let $\mathbb{1}$ denote the all-1's matrix, and let $\mathbb{0}$ denote the zero matrix. Let $Y = V \setminus X \setminus \{v\}$. Define the following matrices:

$$\begin{aligned} L_{11} &= A[X \cap N, Y \cap N], \\ L_{12} &= A[X \cap N, Y \setminus N], \\ L_{21} &= A[X \setminus N, Y \cap N], \text{ and} \\ L_{22} &= A[X \setminus N, Y \setminus N]. \end{aligned}$$

Note that L_{11} is the adjacency matrix of the neighbours of v, L_{12} is the adjacency matrix between neighbours and non-neighbours of v, $L_{21} = L_{12}^T$, and that L_{22} represents the adjacency matrix of non-neighbours of v, relativized to X and Y.

Then

$$\rho_{G/v}(X) = r(A(G/v)[X, Y]).$$

By applying Lemma 4.1.2 we have that

$$\begin{aligned}
r(A(G/v)[X,Y]) &= r\left(\begin{bmatrix} \mathbb{1}+L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}\right) \\
&= r\left(\begin{bmatrix} 1 & \mathbb{0} & \mathbb{0} \\ \mathbb{0}^T & \mathbb{1}+L_{11} & L_{12} \\ \mathbb{0}^T & L_{21} & L_{22} \end{bmatrix}\right) - 1 \\
&= r\left(\begin{bmatrix} 1 & \mathbb{1} & \mathbb{0} \\ \mathbb{0}^T & \mathbb{1}+L_{11} & L_{12} \\ \mathbb{0}^T & L_{21} & L_{22} \end{bmatrix}\right) - 1.
\end{aligned}$$

Now by elementary row operations

$$\begin{aligned}
r\left(\begin{bmatrix} 1 & \mathbb{1} & \mathbb{0} \\ \mathbb{0}^T & \mathbb{1}+L_{11} & L_{12} \\ \mathbb{0}^T & L_{21} & L_{22} \end{bmatrix}\right) &= r\left(\begin{bmatrix} 1 & \mathbb{1} & \mathbb{0} \\ \mathbb{1}^T & L_{11} & L_{12} \\ \mathbb{0}^T & L_{21} & L_{22} \end{bmatrix}\right) \\
&= r\left(\left[\begin{array}{c|c} 1 & A(G)[\{v\}, V(G)\setminus X\setminus\{v\}] \\ \hline A(G)[X,\{v\}] & A(G)[X, V(G)\setminus X\setminus\{v\}] \end{array}\right]\right).
\end{aligned}$$

Hence

$$\rho_{G/v}(X) = r\left(\left[\begin{array}{c|c} 1 & A(G)[\{v\}, (V(G)\setminus X)\setminus\{v\}] \\ \hline A(G)[X,\{v\}] & A(G)[X, (V(G)\setminus X)\setminus\{v\}] \end{array}\right]\right) - 1,$$

as desired. □

Now from Lemmas 3.2.1 and 4.1.3 one can obtain the following result analogous to the Bixby-Coullard Inequality; see Oxley [17, Lemma 8.7.1].

Lemma 4.1.4 (Oum [16]). *Let v be a vertex in a simple graph G. If (C_1, C_2) and (D_1, D_2) are partitions of $V(G)\setminus v$, then the following inequalities hold:*

$$\begin{aligned}
\rho_{G\setminus v}(C_1) + \rho_{G/v}(D_1) &\geq \rho_G(C_1\cap D_1) + \rho_G(C_2\cap D_2) - 1, \\
\rho_{G\setminus v}(C_1) + \rho_{G\circ v}(D_1) &\geq \rho_G(C_1\cap D_1) + \rho_G(C_2\cap D_2) - 1, \text{ and} \\
\rho_{G/v}(C_1) + \rho_{G\circ v}(D_1) &\geq \rho_G(C_1\cap D_1) + \rho_G(C_2\cap D_2) - 1.
\end{aligned}$$

Proof. (due to Oum in [16]) First observe that as $G \circ v = G * w * v * w$ for some neighbour w of v that

$$\rho_{G/v}(C_1) + \rho_{G \circ v}(D_1) = \rho_{(G*v) \setminus v}(C_1) + \rho_{(G*v) \circ v}(D_1)$$

and

$$\rho_{G \setminus v}(C_1) + \rho_{G \circ v}(D_1) = \rho_{(G*w) \setminus v}(C_1) + \rho_{(G*w)/v}(D_1).$$

Hence we only need to show that

$$\rho_{G \setminus v}(C_1) + \rho_{G/v}(D_1) \geq \rho_G(C_1 \cap D_1) + \rho_G(C_2 \cap D_2) - 1.$$

Let $A = A(G)$. Note that for a partition (X, Y) of $V(G) \setminus v$ that:

$$\left[\begin{array}{c|c} 0 & A[\{v\}, C_2] \\ \hline A[C_1, \{v\}] & A[C_1, C_2] \end{array}\right] = A[X \cup \{v\}, Y \cup \{v\}],$$

Hence,

$$\begin{aligned} \rho_{G \setminus v}(C_1) + \rho_{G \circ v}(D_1) &= r(A[C_1, C_2]) + r(A[D_1 \cup v, D_2 \cup v]) - 1 \\ &\geq r(A[C_1 \cap D_1, C_2 \cup D_2 \cup v]) + r(A[C_1 \cup C_2 \cup v, C_2 \cap D_2]) - 1 \\ &= \rho_G(C_1 \cap D_1) + \rho_G(C_2 \cap D_2) - 1. \end{aligned}$$

□

4.2 Vertex Minors and Splits

Using Lemma 4.1.4 we obtain the following result which is analogous to Lemma 8.7.3 in [17]. We first introduce the analogue of internally 3-connected graphs for primality; a graph is *internally prime* if the only splits it has are of the form (A, B) where $\min(|A|, |B|) \leq 2$.

Lemma 4.2.1. *Let G be a prime graph, and $v \in V(G)$. Then two of $G \setminus v$, $G \circ v$, and G/v are internally prime.*

Proof. Suppose not for a contradiction. Then two of $G \setminus v$, G/v, and $G \circ v$ are not internally prime. By possibly replacing G by $G * w$ for some neighbour w of v we may assume that neither $G \setminus v$ nor G/v are internally prime. Hence we have splits (C_1, C_2) and (D_1, D_2) of $G \setminus v$ and G/v respectively with $|C_i| \geq 3$, and $|D_i| \geq 3$. Now, by Lemma 4.1.4,

$$2 = \rho_{G \setminus v}(C_1) + \rho_{G/v}(D_1) \geq \rho_G(C_1 \cap D_1) + \rho_G(C_2 \cap D_2) - 1,$$

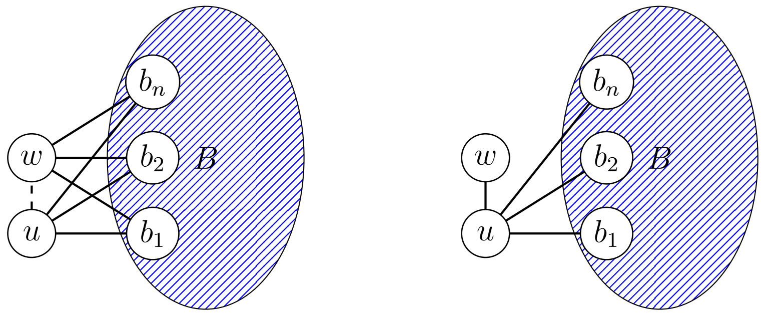

Figure 4.1: Small Splits in a Connected Graph

hence either $\rho_G(C_1 \cap D_1) \leq 1$ or $\rho_G(C_2 \cap D_2) \leq 1$. However, as G is prime, we have that one of $C_1 \cap D_1$, $D_2 \cup C_2 \cup \{v\}$, $C_2 \cap D_2$, or $C_1 \cup D_1 \cup \{v\}$ has size at most one. As $|C_i| \geq 3$, and $|D_i| \geq 3$, we have that either $C_1 \cap D_1$ or $C_2 \cap D_2$ has at most one vertex. Symmetrically,

$$2 = \rho_{G \setminus v}(C_1) + \rho_{G/v}(D_2) \geq \rho_G(C_1 \cap D_2) + \rho_G(C_2 \cap D_1) - 1,$$

so either $C_1 \cap D_2$ or $C_2 \cap D_1$ has at most one vertex. In any case, one of C_1, C_2, D_1, D_2 has at most two vertices, a contradiction, as desired. $\square$

The following is a useful observation on small splits in a connected graph.

Lemma 4.2.2. *Let G be a connected graph, $(\{u, w\}, V(G) \setminus \{u, w\})$ be a split. Then either*

- $\deg(u) = 1$*, and u is adjacent to w, or*
- $\deg(w) = 1$*, and u is adjacent to w, or*
- $N(u) \cap (V(G) \setminus \{u, w\}) = N(w) \cap (V(G) \setminus \{u, w\})$*.*

Proof. Direct from the definition of a split; see Figure 4.1. $\square$

Lemma 4.2.3. *Let v be a vertex in a simple prime graph G. If $G \setminus v$ is internally prime with a split $(\{u, w\}, V(G) \setminus \{u, v, w\})$, then there exists a graph G' that is locally equivalent to G or $G * u$ such that*

- *u has degree one in $G' \setminus v$, and*

- *u is adjacent to v in G'.*

Proof. Let $(\{u, w\}, B)$ be a split of $G \setminus v$. By Lemma 4.2.2 either u or w has degree 1 in $G \setminus v$ or $N_{G\setminus v}(u) \cap B = N_{G\setminus v}(w) \cap B$. If either u or w has degree one in $G \setminus v$ we're done by taking $G' = G$, so both u and w have degree at least two in $G \setminus v$, and that $N_{G\setminus v}(u) \cap B = N_{G\setminus v}(w) \cap B$. Since $(\{u, w\}, B \cup \{v\})$ is not a split of G, we have that exactly one of u or w is adjacent to v in G. We may assume that u is by symmetry.

Furthermore, as $(\{u, v, w\}, B)$ is not a split of G either there exists a vertex $x \in (N_G(w) \cap B) \setminus N_G(v)$ or $N_G(w) \cap B \subseteq N_G(v) \cap B$.

If the former we may possibly locally complement at x to obtain a graph G'' where u is adjacent to w.

If the latter if w is not adjacent to u we will locally complement at u to obtain a graph where there is a vertex in $N(w)$ not in $N(v)$. Now we may possibly locally complement at x to obtain a graph G'' where u is adjacent to w. Note that w is not adjacent to v in G'', as we only complemented if w was not adjacent to u.

In any case we obtain a graph where u is adjacent to w. Finally, by locally complementing at w in G'' we get a graph G' where:

- G' is locally equivalent to G.
- u has degree one in $G' \setminus v$, and,
- u is adjacent to v in G',

as desired. □

4.3 Building a Prime Graph

We would like to understand the structure of prime graphs under taking vertex minors. To that end, we need one technical definition which captures the structure that one has when one cannot remove a vertex up to local equivalence and stay prime. Consider a graph of the form depicted in Figure 4.2; note that none of $G \setminus c_1$, $G \circ c_1$, and G/c_1 is prime. As it turns out, Figure 4.2 captures up to local equivalence when a vertex cannot be removed while preserving primality. Hence we say an *envelope* of a graph G is a five-tuple $(c_1, c_2, f_3, f_2, f_1)$ such that $N(c_1) = \{c_2, f_1\}$, $N(c_2) = \{c_1, f_3\}$, and $N(f_2) = \{f_1, f_3\}$. We say c_1 and c_2 are

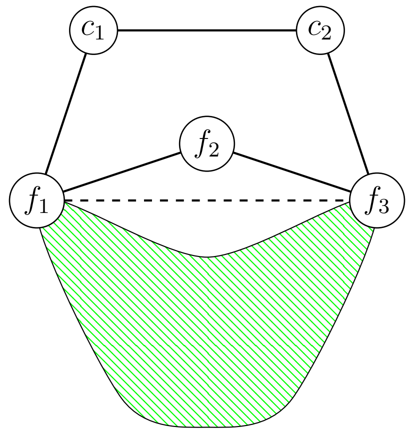

Figure 4.2: An Envelope

the *corners* of the envelope and f_1, f_2, and f_3 are the *flaps* of the envelope, with f_2 being the *center flap* of the envelope.

Note that a 1-separation (A, B) with common vertex v induces a split $(A \setminus \{v\}, B)$, and hence prime graphs have no cut vertices.

Let v be a vertex in a simple graph G. We say that G' is *locally v-equivalent* to G if G' can be obtained from G by a sequence of local complementations on vertices in $V(G) \setminus (\{v\} \cup N(v))$. Note that if G' and G are locally v-equivalent graphs then $G\square v$ is locally equivalent to $G'\square v$ for $\square \in \{\setminus, \circ, /\}$; this follows immediately from Lemma 3.1.2(1).

The following is a new generalization of Lemma 4.0.1. Note that we have used the fact that if a vertex v is not removable in a way that preserves primality then it is in an envelope extensively in Chapter 2. We will see later that three triangles of R that share a common edge corresponds to an envelope of $\mathrm{IG}(R, T)$.

Lemma 4.3.1. *Let v be a vertex in a prime graph G. If neither $G \setminus v$, $G \circ v$, nor G/v are prime, then v is a corner of some envelope F, up to local equivalence.*

Proof. From Lemma 4.2.1 we may assume without loss of generality that $G \setminus v$ is internally prime. If $G \setminus v$ is prime then we are done. Now if $G \setminus v$ is not prime there are two vertices u and w such that $(\{u, w\}, V(G) \setminus \{u, w\})$ is a split in $G \setminus v$. By Lemma 4.2.3 we have a graph H locally equivalent to G such that v is adjacent to u, and $\deg_H(u) = 2$.

Consider now $H * v * u * v = H'$. Note that v has degree two in H', and is only adjacent to u and w. By Lemma 4.2.1 at least one of $H' \setminus v = H/v$ or $H' \circ v = H \circ v * u$ is internally prime. Let $H'' = H'$ if $H' \setminus v$ is internally prime and let $H'' = H' * v$ if $H \circ v$ is internally prime. Note that $H'' \setminus v$ is locally equivalent to one of $H' \setminus v$ or $H' \circ v$ and $H'' \circ v$ is locally equivalent to the other. By the following claim we may assume that there is no edge between u and w in H'' and $H'' \setminus v$. Note that $H'' \setminus v$ is internally prime.

Claim 4.3.2. *There exists a graph H''' locally v-equivalent to H'' where u is not adjacent to w.*

Proof. As H''' is prime, it has no cut vertices, as a cut vertex induces a split of H'''. Proceed by induction on the length n of the shortest path $P = ux_1x_2 \dots x_nw$ in H'' avoiding v and the possible $\{u, w\}$ edge. If $n = 1$ we may locally complement by x_1. Inductively, locally complement by x_1 and proceed by induction on the new shortest path $P' = ux_2 \dots x_nw$. □

Now if $H'' \setminus v$ is prime we are done. Otherwise, there is a nontrivial split (A, B) of $H'' \setminus v$, with $|A| = 2$, $|B| \geq 2$. As H'' is prime exactly one of $u, w \in A$. Without loss of generality we may assume that $u \in A$. Let z be the other vertex in A. By Lemma 4.2.3 we have that there is some graph H''' in which u has degree two and is only adjacent to z and v, with H''' locally equivalent to H''. Note that $H''' \setminus v$ is internally prime.

Now consider H'''/v; this graph has the same edge set as $H''' \setminus v$ with the single difference being that u is now adjacent to w in H'''/v.

Claim 4.3.3. *The graph H'''/v is internally prime.*

Proof. Suppose not for a contradiction; let (A, B) be a split with $|A| \geq 3, |B| \geq 3$. As $H''' \setminus v$ is internally prime, we know that the edge $\{u, w\}$ is in $\delta_{H''' \circ v}(A)$. Without loss of generality we may assume that $u \in A$ and $w \in B$. However, u only has degree two, so $\delta(A) \subseteq \delta(u)$.

Moreover, $\delta(A) = \delta(u)$, as otherwise w or z would be a cut vertex. Hence $z \in B$. Let c and d be the two other vertices in A. As v is adjacent to neither c nor d, $N_{H''' \circ v}(\{c, d\}) = N_{H'''}(c, d)$. Hence $(\{c, d\}, V(H''') \setminus \{c, d\})$ gives a split of H''', a contradiction, as desired. □

If $H''' \circ v$ is prime we are done. Otherwise, it remains to show that up to local equivalence, v is a corner of an envelope of G, up to local equivalence.

Claim 4.3.4. *There exists an envelope F of H''' such that v is a corner of F.*

Proof. From above we know that $H''' \circ v$ is not prime, hence there is a nontrivial split (A, B) of $H''' \circ v$, with $|A| = 2, |B| \geq 2$. As H''' is prime, at least one of u, w is in A. As G''' is prime we know that $N_{H'''}(w) \setminus v$ is nonempty, and hence we cannot have both u, w in A.

Suppose for a contradiction that $w \in A$. Hence the edge $\{u, w\}$ in $\delta(H''' \circ v)(A)$ and therefore so is the edge $\{u, z\}$. Hence $z \in A$. However, w and z have distinct neighbour sets in $H''' \circ v$, as H''' is prime. Hence (A, B) is not a split of $H''' \circ v$, a contradiction, as desired.

Hence $u \in A$; let a be the other vertex in S. Now a is adjacent only to u and w, hence $\{w, u, v, a, z\}$ forms an envelope in H''', with w, a, z as the flap, and u, v as the corners. □

Hence v is, up to local equivalence, a corner of some envelope F of G, as desired. □

Note that the center flap in an envelope can be removed in two ways preserving primality unless G is locally equivalent to C_5.

Lemma 4.3.5. *Suppose $(c_1, c_2, f_3, f_2, f_1)$ is an envelope of a simple graph G. If G is not locally equivalent to C_5, then both $G \setminus f_2$ and G/f_2 are prime.*

Proof. Suppose not for a contradiction. Then either $G \setminus f_2$ or G/f_2 has a split (A, B) with $|A| \geq 2$ and $|B| \geq 2$. Note that these two graphs are otherwise identical except for the presence of the $\{f_1, f_3\}$ edge. By possibly taking $G = G * f_2$ we will assume that $\{f_1, f_3\}$ is not an edge in $G \setminus f_2$ and is an edge in G/f_2. We take these two cases in turn.

Case 1: *$G \setminus f_2$ is not prime.* As G is prime, f_1 and f_3 are in different parts of the partition (A, B). We may assume that $f_1 \in A$ and $f_3 \in B$. Now $\{f_1, c_1, c_2, f_3\}$ is a path that crosses the split. Hence one of $\{f_1, c_1\}$, $\{c_1, c_2\}$, or $\{c_2, f_3\}$ is in $\delta(A)$.

Case 1.1: *$\{f_1, c_1\}$ is in $\delta(A)$.* Hence $c_1 \in B$. Now every other vertex in A that is not f_1 either has no neighbours in B or is adjacent to c_1. Note that the only other vertex adjacent to c_1 is c_2. Now $c_2 \notin A$ as $f_3 \in B$ and f_1 is not adjacent to f_3. Hence every other vertex in A has no neighbours in B. Hence f_1 is a cut-vertex in $G \setminus f_2$. As $N_G(f_2) \cap A = \{f_1\}$, f_1 is a cut vertex in G, a contradiction, as desired.

Case 1.2: *$\{c_1, c_2\}$ is in $\delta(A)$.* As $\{f_1, c_1\} \notin \delta(A)$, $c_1 \in A$, and $c_2 \in B$. Now there is no other vertex in A other than c_1 with a neighbour in B, and similarly there is no other vertex in B other than c_2 with a neighbour in A. Hence f_1 and f_3 are cut vertices unless $|A| = 2$ and $|B| = 2$. Now f_1 and f_3 are not cut vertices as they would lift to cut vertices in G, as $N_G(f_2) = \{f_1, f_3\}$. Hence $|A| = 2$, and $|B| = 2$. Hence $A = \{f_1, c_1\}$ and $B = \{f_3, c_2\}$. Hence $G \setminus v$ is a path on four vertices and thus G is a five-cycle, as desired.

Case 1.3: $\{c_2, f_3\}$ *is in* $\delta(A)$. This case is symmetric to Case 1.1.

Case 2: G/f_2 *is not prime.* As G is prime, f_1 and f_3 are in different parts of the partition (A, B). We may assume that $f_1 \in A$ and $f_3 \in B$. If $c_1 \in A$, then $c_2 \in A$, as f_1 is not adjacent to c_2, but is adjacent to f_3. Now every other vertex in B that is not f_3 has no neighbours in A, as no other vertex except c_1 and f_3 is adjacent to c_2. Hence f_3 is a cut vertex in G/f_2, and hence in G, a contradiction.

Hence $c_1 \in B$. By a symmetric argument, $c_2 \in A$. Now every other vertex in A that is not f_1 has no neighbours in B, and likewise every other vertex in B that is not f_3 has no neighbours in A, as $N_{G/f_2}(c_1) = \{c_2, f_1\}$) and $N_{G/f_2}(c_2) = \{f_3, c_1\}$. Hence if $|A| \geq 3$ or $|B| \geq 3$, f_1 or f_3 is a cut-vertex of G/f_2, and hence G as $N_G(f_2) = \{f_1, f_3\}$. Hence $|A| = |B| = 2$ and $G/f_2 = C_4$, and hence $G = C_5$. □

Futhermore, all three ways of removing a corner of an envelope of a prime give a graph that is internally prime.

Lemma 4.3.6. *Suppose* $(c_1, c_2, f_3, f_2, f_1)$ *is an envelope of a simple graph* G. *Then all three ways of removing* c_1 *and* c_2 *up to local equivalence are internally prime.*

Proof. Suppose not. As $G \circ c_1$ is isomorphic to $G \setminus c_2 * f_2$, we need only consider $G \setminus c_1$ and G/c_1. We take these two cases in turn.

Case 1: $G \setminus c_1$ *is not internally prime.* Let (A, B) be a split with $|A| \geq 3$ and $|B| \geq 3$. As G is prime, we have that f_1 and c_2 are in different parts of the partition (A, B). Hence we may assume without loss of generality that $f_1 \in A$ and $c_2 \in B$. Now $f_3 \in B$ as if $f_3 \in A$ then no other vertex is in B as every vertex in B would have to have f_3 as its sole neighbour, contradicting the fact that G was prime. Now $f_2 \in A$, as if $f_2 \in B$ then f_1 would be adjacent to c_2, which is not the case. Consider the third vertex in B. Such a third vertex would be adjacent to f_2, as $f_2 \in A$ is adjacent to $f_3 \in B$. However, f_2 has no other neighbours except f_1 and f_3, a contradiction, as desired.

Case 2: G/c_1 *is not internally prime.* Let (A, B) be a split with $|A| \geq 3$ and $|B| \geq 3$. As G is prime, we have that f_1 and c_2 are in different parts of the partition (A, B). Hence we may assume without loss of generality that $f_1 \in A$ and $c_2 \in B$. By locally complementing at f_2 we may assume that f_3 is not adjacent to f_1.

Claim 4.3.7. $f_3 \in A$.

Proof. Suppose not. Then $f_2 \in B$ as $c_2 \in B$ is not adjacent to f_2 but f_3 is. Now every vertex in A is either adjacent to nothing in B or has c_2 and f_2 as neighbours, as $f_1 \in A$ is has c_2 and f_2 as neighbours. As $f_3 \in B$, no other vertex in A other than f_1 is adjacent to anything in B. Hence f_1 is a cut vertex of G/c_1. As c_1 is only adjacent to f_1 and c_2, f_1 is a cut vertex of G, contradicting the fact that G is prime, as desired. $\square$

Now every other vertex in B that is not c_2 is either adjacent to exactly f_1 and f_3 or it is not adjacent to anything in A. As G is prime, f_2 is the only other vertex in B adjacent to exactly f_1 and f_3. Hence every other vertex in B that is neither f_2 nor c_2 is isolated in G/c_1. As c_1 is only adjacent to f_1 and c_2 in G, those vertices are isolated in G. Hence B contains no other vertices except c_1 and possibly f_2, contradicting the assumption that $|B| \geq 3$. $\square$

As a corollary we obtain the following result which generalizes Bouchet's decomposition theorem for prime graphs [3].

Corollary 4.3.8. *Let G be a prime graph that is not locally equivalent to C_5. Then for each vertex v in G, either*

- *one of $G \setminus v$, $G \circ v$, or G/v is prime, or*
- *each of $G \setminus v$, $G \circ v$, and G/v is internally prime, and there is a vertex $w \in V(G)$ such that two of $G \setminus w$, $G \circ w$, and G/w are prime.*

4.4 Internally Six-Edge Connected Tour Graphs

We finish this chapter with a few remarks on how primality relates to connectivity in the tour graph. Recall that a four-regular graph is *weakly* six-edge-connected if it is four-edge connected and every cut on four edges partitions the graph into two components, one of which has size at most two.

Lemma 4.4.1. *Let R be a tour graph for a circle graph G. If G is prime, then R is internally six-edge-connected. Similarly, if G is internally prime, then R is weakly six-edge-connected.*

Proof. We will prove that if G is prime then R is internally six-edge-connected; the proof is analogous for internal primality. Suppose R is not internally six-edge-connected. Then there is a four-edge cut $\{e, f, g, h\}$ of R that partitions the vertices of R into two sets A and B with $|A|, |B| \geq 2$. Let T be the tour of R corresponding to $\mathcal{C}$. We may assume without loss of generality by possibly renaming edges and reversing T that T starts in A, goes through e then f then g then h. Hence the chord diagram $\mathcal{C}$ is of the form shown in Figure 4.3. Therefore we have that (A, B) is a split of G, as desired. $\square$

Lemma 4.4.2 (Bouchet [1]). *Let R be a tour graph for a circle graph G. If R is internally six-edge-connected, then G is prime.*

Proof. Suppose for a contradiction that G is not prime. We may assume without loss of generality that G is connected as the result follows directly if G is disconnected. Then there is some split (A, B) of G with $|A| \geq 2$, and $|B| \geq 2$. Now A and B partition the circumference of $\mathcal{C}$ into intervals $(A_i : i \in \mathbb{Z}/2k\,\mathbb{Z})$ and $(B_j : j \in \mathbb{Z}/2k\,\mathbb{Z})$ which contain either chords in A or chords in B. We may assume without loss of generality that the intervals appear along the circle in clockwise order starting from A_0, B_0 and ending with A_{2k-1}, B_{2k-1}. Let $M(I)$ denote the set of intervals J such that there is a chord with an end in both I and J.

Claim 4.4.3. *If A_j and A_k are in $M(A_i)$ with $j \neq i$ and $k \neq i$, then $j = k$. Symmetrically, if B_j and B_k are in $M(B_i)$ with $j \neq i$ and $k \neq i$ then $j = k$.*

Proof. Suppose not for a contradiction. We may assume $i = 0$ by relabeling. Then there is a chord a from A_0 to A_j and a chord a' from A_0 to A_k with $0 < j < k$, where $<$ is the natural order over $\mathbb{Z}$. Now consider a chord b in B_l with $j \leq l \leq k$ that does not have its other end in B_l; such a chord exists as G is connected. Now b can't cross both a and a', but it must cross one, contradicting the assumption that (A, B) is a split, as desired. $\square$

Now since all the A-chords cross all the B-chords, we have that chords with an end in A_i have their other end in either A_i or A_{i+k} and similarly chords in B_i have their other end in either B_i or B_{i+k}. Now if any of the A_i have at least two chords we are done; just take the four arcs of the circle incident to it and the arc A_{i+k}; this gives a four-edge cut in R. A symmetric argument works if any of the B_i have at least two chords. Otherwise each interval has only one chord end in it. Thus $\mathcal{C}$ consists of k chords which each pairwise cross, and hence R is a cycle on at least four vertices in which every edge has been replaced by a parallel pair. Hence R has a four-edge cut with at least two vertices on each side, as desired. $\square$

From Lemmas 4.4.1 and 4.4.2 we can obtain an analogue of Lemma 4.3.1 for four-regular tour graphs. We need a few propositions however.

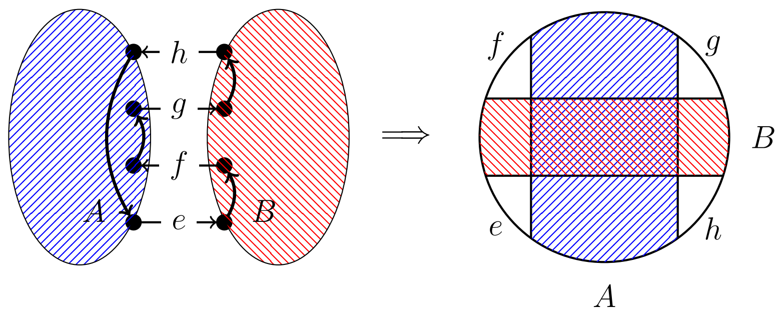

Figure 4.3: R, T, $\mathcal{C}$, and a four-edge cut of R.

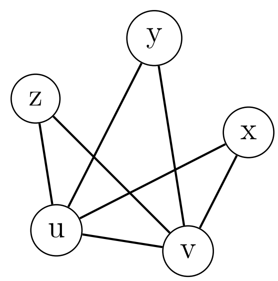

Figure 4.4: Edge in Three Triangles.

Lemma 4.4.4. *Let G be a prime circle graph and let R be the tour graph for G. If G contains an envelope $B = (u, v, x, y, z)$ where u, v are the corners and y is the middle vertex in the flap of B, then $\{u, v\}$ is an edge contained in three triangles of R.*

Conversely, if $\{u, v\}$ is an edge in three triangles of R then G (up to local equivalence) contains an envelope, where the u and v are the corners of that envelope.

Proof. Consider a chord diagram representation $\mathcal{C}$ for G. The adjacencies in the envelope show that vertices are traversed in the Euler tour in the cyclic order $(y, u, z, v, u, x, v, y, z, x)$ or its reverse.

Conversely, suppose $\{u, v\}$ is an edge of three triangles of R. Let $\{x, u, v\}$, $\{y, u, v\}$, and $\{z, u, v\}$ be those three triangles, as shown in Figure 4.4.

As R is internally six-edge-connected deleting the edges in $\delta(V(R) \setminus \{u, v, x, y, z\}) \cap (\delta(x) \cup \delta(z))$ does not disconnect the graph. Hence we may find a tour $\mathcal{C}$ of R such that the transition from z is from $\{z, u\}$ to $\{z, v\}$ and the transition at x is from $\{u, x\}$ to $\{u, v\}$.

Now $\mathcal{C}$ is of the form $\ldots Q \ldots$, where Q is a tour of those three triangles which starts and ends at y. Hence $\mathcal{C}' = \ldots yuzvuxvy \ldots$ is also a valid tour for R. Therefore from above we have that the circle graph G' corresponding to $\mathcal{C}'$ contains an envelope where the corners are u and v, as desired. $\square$

Hence as a consequence of Lemmas 4.3.6, 4.3.5, 4.4.4, 4.4.1, and 4.4.2 we obtain the following versions of Lemmas 4.3.1 and 4.2.1 for four-regular graphs.

Lemma 4.4.5. *Let v be a vertex in an internally six-edge connected four-regular graph R. Then two out of the three ways to split off v in R result in weakly-six-edge connected graphs.*

Lemma 4.4.6. *Let v be a vertex in an internally six-edge connected four-regular graph R. Either there is a way to split off v in R while remaining internally-six-edge connected or v is incident to an edge in three triangles of R.*

Moreover, if R is not isomorphic to K_5, and $\triangle_1$, $\triangle_2$, and $\triangle_3$ are three triangles of R that share an edge, then for all i and j in $\{1, 2, 3\}$, there are two ways to split off at the single vertex in $V(\triangle_i) \setminus V(\triangle_j)$ while remaining internally six-edge connected.

We wrap up this chapter with two easy remarks on small, internally six-edge connected graphs.

Lemma 4.4.7. *If R is a simple four-regular graph on at most nine vertices, then R is internally six-edge connected if and only if R is K_4-subgraph-free.*

Proof. If R contains a K_4 subgraph H then $|\delta(H)| \leq 4$ as R is four-regular. Conversely, suppose that R contains a cut S with $|S| \leq 4$ which partitions $V(R)$ into two parts (A, B). As R is simple we have that $|A| \geq 4, |B| \geq 4$, and hence one of $|A|, |B|$ is exactly four. We may assume without loss of generality that $|A| = 4$. As R is simple, it follows that $R[A] \cong K_4$, and $|S| = 4$. $\square$

Lemma 4.4.8. *The complement of the cube is the only graph on eight vertices that is internally six-edge connected.*

Proof. Let R be a simple four-regular graph on eight vertices that is not internally six-edge connected. From Lemma 4.4.7, there is an edge cut S of R with $|S| = 4$. As R is simple, S partitions $V(R)$ into two parts (A, B) with $|A| = |B| = 4$, and hence $R[A] \cong R[B] \cong K_4$. Now up to isomorphism there is exactly one way to place the edges in S, hence G is unique, as desired. $\square$

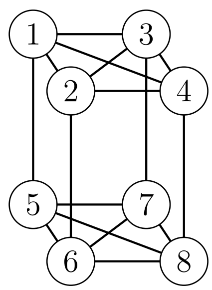

Figure 4.5: Complement of the Cube

Chapter 5

Extended Representations

In this section we give a representation, previously presented in a different form by Bouchet in [1] for single vertex extensions of a circle graph G represented by a chord diagram $\mathcal{C}$. When the graph G is prime, we often omit $\mathcal{C}$ due to unique representability.

5.1 Extended Chord Diagrams

Recall a *hyperchord* Σ is an even set of arcs of a chord diagram $\mathcal{C}$. An arc is *even* if it is not in the hyperchord, and *odd* otherwise. An extended chord diagram $(\mathcal{C}, \Sigma)$ consists of a chord diagram $\mathcal{C}$ and hyperchord Σ of C. An *extended circle graph* G for an extended chord diagram $\mathrm{IG}(\mathcal{C}, \Sigma)$ is constructed from a circle graph G' of $\mathcal{C}$ by adding a new vertex v for the hyperchord Σ, where v is adjacent to $c \in V(G')$ if and only if c partitions the arcs of Σ into two odd parts. Local complementations on extended circle graphs can be easily described in terms of their extended representations so long as it is not the hyperchord being complemented. That is, two extended chord diagrams $\mathrm{IG}(\mathcal{C}_1, \Sigma_1)$ and $\mathrm{IG}(\mathcal{C}_2, \Sigma_2)$ are *locally equivalent* if the underlying chord diagrams are and $\Sigma_1 = \Sigma_2$. Now the following lemma gives us a characterization of local equivalence classes of extended chord diagrams.

Lemma 5.1.1. *Let* $H = \mathrm{IG}(\mathcal{C}, \Sigma)$. *Let* c *be any chord of* $\mathcal{C}$. *Then* $\mathrm{IG}(\mathcal{C} * c, \Sigma) = H * c$.

Proof. Fix a chord $c \in \mathcal{C}$, and let v be the extension vertex. We present the case when v is adjacent to c; the other case is symmetric. As v is adjacent to c, c partitions the arcs A of $\mathcal{C}$ into two parts A_1, A_2, each containing an odd number of arcs of Σ. Let d be chord in $\mathcal{C}$ that is not c; it partitions A into two parts A_3, A_4. Consider the following two cases.

Case 1: *d is not adjacent to c.* We may assume without loss of generality that $A_3 \subset A_1$. Now locally complementing at c preserves the partition that d induces, as $A_3 \subset A_1$. Hence d is adjacent to v in $H * c$ if and only if it is adjacent to v in H, as desired.

Case 2: *d is adjacent to c.* As d is adjacent to v, $|A_3|$ and $|A_4|$ are odd. As c is adjacent to v, we have that one of $|A_1 \cap A_3|$, $|A_1 \cap A_4|$ is odd, and one of $|A_2 \cap A_3|$, $|A_2 \cap A_4|$ is odd. Without loss of generality we may assume that $|A_1 \cap A_3|$ and $|A_2 \cap A_4|$ is odd. Now locally complementing at c switches the partition that d induces to $((A_2 \cap A_4) \cup (A_1 \cap A_3), (A_2 \cap A_3) \cup (A_1 \cap A_4))$, which has different parity than (A_3, A_4). Hence d is adjacent to v in $H * c$ if and only if it is not adjacent to v in H, as desired. □

Chord deletion also has a straightforward extension to an extended chord diagram; we identify two arcs together preserving parity, so two odd arcs and two even arcs are identified down to a single even arc, and an odd and even arc are identified to a single odd arc.

The following lemma illustrates why extended chord diagrams are useful when working with single vertex extensions of circle graphs.

Lemma 5.1.2 (Bouchet [1]). *Let H be a single vertex extension of a circle graph G with some chord diagram $\mathcal{C}$. Then there are $2^{|V(G)|-1}$ unique hyperchords Σ for $\mathcal{C}$ such that $\mathrm{IG}(\mathcal{C}, \Sigma) = H$.*

Proof. Let C be the set of chords for $\mathcal{C}$, and fix some chord $c_0 \in C$. Pick an arbitrary end of c_0 to be the *head* of c, and the other end to be the *tail* of c_0. Now label the chords of C starting at the head c_0 going clockwise by $c_1, c_2, \ldots c_n$ and label the arcs of $\mathcal{C}$ by $a_1, a_2, \ldots a_{2n}$ starting at c_0 going clockwise. We denote the *head* of a chord c_i for $i > 0$ to be the end of a chord c_i first encountered from a clockwise walk from the head of c_0, and the *tail* of c_i to be the other end.

For each chord c_i define I_i to be the set of arcs a_i with a_i contained in the closed segment defined by a clockwise walk from the head of c_i to the tail of c_i. Observe that the last arc a_{2n} is not contained in any I_i, as it is encountered after the tail of any chord c on a clockwise walk starting at c. Consider the following $|C| \times 2|C|$ matrix A over GF(2), with columns indexed by the arcs a_j and rows indexed by the chords c_i:

$$A_{c_i,a_j} = \begin{cases} 1, & \text{if } c_j \in I_i \\ 0, & \text{otherwise.} \end{cases}$$

This matrix has full row rank, as for every chord c_i there is some arc a_k present in I_i but not present in any I_j for $j > i$; namely the arc which occurs immediately after the head of

c_i. Furthermore, as the column $A_{a_{2n}}$ is an all-zeros column, the $(|C|+1) \times 2|C|$ matrix B obtained by adding an all-1's row:

$$B = \left[\begin{array}{c} \mathbb{1}^T \\ \hline A \end{array}\right]$$

has full row rank.

Let v be the single vertex in $V(H) \setminus V(G)$. Let b be a column vector indexed by the chords c_i in the following manner:

$$b_{c_i} = \begin{cases} 1, & \text{if } c_i \text{ is adjacent to } v \text{ in } H \\ 0, & \text{otherwise.} \end{cases}$$

Now consider a solution $\overline{x}$ to the following system of linear equations over GF(2).

$$\left[\begin{array}{c} \mathbb{1}^T \\ \hline A \end{array}\right] x = \left[\begin{array}{c} 0 \\ \hline b \end{array}\right] \tag{5.1}$$

As B has full column rank, there exists such a solution $\overline{x}$. Moreover, the dimension of the solution space is $2n - (n+1) = n-1$, giving rise to 2^{n-1} possible solutions. Let $\Sigma = \{a_i : \overline{x}_{a_i} = 1\}$. As $\overline{x}$ has even support, $|\Sigma|$ is even. Furthermore, for every chord $c \in C$,

c is adjacent to v if and only if c partitions Σ into two odd parts.

Hence H is an extended circle graph with representation $(\mathcal{C}, \Sigma)$ as desired. □

As the arcs of a chord diagram $\mathcal{C}$ correspond to the edges of its tour graph R, we now explicitly construct a correspondence between *signed graphs* [10] and hyperchords that was first implicitly introduced by Bouchet in [1]. Recall a *signed graph* (G, Σ) is a graph G along with a $\Sigma \subseteq E(G)$. Edges are *even* if they are not in Σ, *odd* otherwise. An *even* signed graph is one where $|\Sigma| \equiv 0 \pmod 2$. Two signed graphs $(G_1, \Sigma_1$ and (G_2, Σ_2) are equivalent if $G_1 = G_2$ and there exists some cut $C \subseteq E(G_1)$ of G_1 with $\Sigma_1 \Delta C = \Sigma_2$.

We first make the following observation:

Lemma 5.1.3. *Let $G = \mathrm{IG}(\mathcal{C}, \Sigma)$, and let R be the tour graph for $\mathcal{C}$. If S is a cut of R, then* $\mathrm{IG}(\mathcal{C}, \Sigma\Delta S) = G$

Proof. Fix an Eulerian tour C of $\mathcal{C}$. Let v be the extension vertex of G. Let c be a chord in $\mathcal{C}$. Observe that C decomposes into cC_1cC_2 for closed walks $C_1, C_2 \subseteq C$. Now, v is adjacent to c if and only if $|C_1 \cap \Sigma|$ and $|C_2 \cap \Sigma|$ are odd. As a cut intersects a cycle, and therefore closed walks, in even parity, we have that $|C_1 \cap \Sigma\Delta S|$ and $|C_2 \cap \Sigma\Delta S|$ have the same parity as $|C_1 \cap \Sigma|$ and $|C_2 \cap \Sigma|$, as desired. □

Hence, by Lemma 5.1.3, for a given extended chord diagram $\mathrm{IG}(\mathcal{C}, \Sigma)$, the equivalence class of signed graphs gives rise to a set of solutions of the linear system described in Lemma 5.1.2. As the dimension of the cut-space of G is $|V(G)| - 1$, there are $2^{|V(G)|-1}$ signed graphs in the equivalence class, so we obtain the following corollary.

Corollary 5.1.4. *Let* $\mathrm{IG}(\mathcal{C}, \Sigma) = H$*, and let* R *be the tour graph for* $\mathcal{C}$*. Then* $\mathrm{IG}(\mathcal{C}, \Sigma') = H$ *if and only if* $\Sigma \Delta \Sigma'$ *is a cut of* R*.*

5.2 Characterizing Obstructions

When a hyperchord Σ has size two, we observe that it is simply a regular chord and the extended chord diagram $(\mathcal{C}, \Sigma)$ is simply a regular chord diagram. Hence we obtain the following useful lemma as a consequence of Lemma 1.4.3.

Lemma 5.2.1. *Let* H *be a prime single vertex extension of a circle graph with representation* $\mathrm{IG}(\mathcal{C}, \Sigma)$*, and let* R *be the tour graph of* C*. Then* H *is a circle graph if and only* (R, Σ) *has a signature of size at most* 2*.*

Now the following key observation illustrates why signed graphs are an useful representation for single vertex extensions of circle graphs.

Lemma 5.2.2. *Let* (R, Σ) *be a signed graph. If* (R, Σ) *has* n *edge-disjoint odd circuits, then every signature of* (R, Σ) *has size at least* n*.*

Proof. As odd circuits are invariant under resigning – cuts intersect circuits in even parity, each odd circuit contains at least one odd edge. $\square$

Chapter 6

Bouchet's Theorem

We will now prove Bouchet's Theorem by combining the reductions proven in Chapter 5 with the new structural results proven in Chapter 2.

First observe that since W_5 admits an extended representation $(K_5, E(K_5))$, which will always have four odd edges in any signature, and since W_7 and W_7 admit representations (R, Σ) with four odd circuits, by Lemma 5.2.1 we have that neither W_5, W_7 nor F_7 are circle graphs.

Now let G be an excluded minor; note that it is a prime graph. By Corollary 4.3.8 we have that there exists a vertex v such that one of G/v, $G \circ v$, or $G \setminus v$ is prime. By renaming G we may assume that $G \setminus v$ is prime.

Now $G \setminus v$ is a prime circle graph; hence by Lemma 1.4.3 there is an unique tour graph R and tour T such that $\mathrm{IG}(R, T) = G \setminus v$. Now G admits a extended representation as a signed four-regular tour graph with tour $\mathrm{IG}(R, \Sigma, T)$. By Lemma 5.2.1 (R, Σ) cannot be resigned to two odd edges and hence by Lemma 2.1.7 we have that R has an odd-K_5 immersion minor or four edge-disjoint odd circuits.

By Lemma 2.2.19 we have that if R has four-edge-disjoint odd circuits then (R, Σ) admits an $R(F_7)$ or $R(W_7)$ immersion minor. Now $R(F_7)$ is a signed-tour graph representation of F_7 and likewise $R(W_7)$ is a signed-tour graph representation of W_7. Hence we have that (R, Σ) has a odd-K_5, $R(F_7)$ or $R(F_7)$ immersion minor. As (R, Σ) is immersion-minor-minimal with respect to being internally-six-edge connected and with respect to not having a signature of size two or fewer by Lemma 5.2.1 we have that R is equivalent to odd-K_5, $R(F_7)$, or $R(W_7)$.

Now odd-K_5 is a signed-tour-graph representation of W_5, and likewise $R(F_7)$ for F_7 and $R(W_7)$ for W_7. Hence we have that G is either W_5, W_7, or F_7.

Hence the set of excluded minors for the class of circle graphs is exactly $\{W_5, W_7, F_7\}$, as desired.

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