Circle Graph Obstructions

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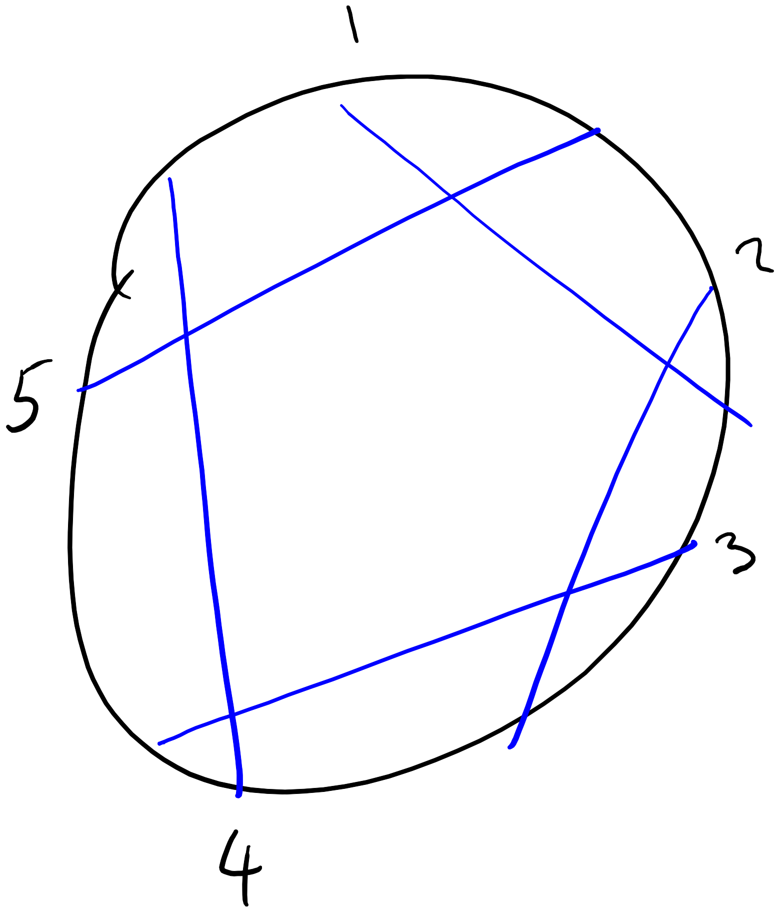
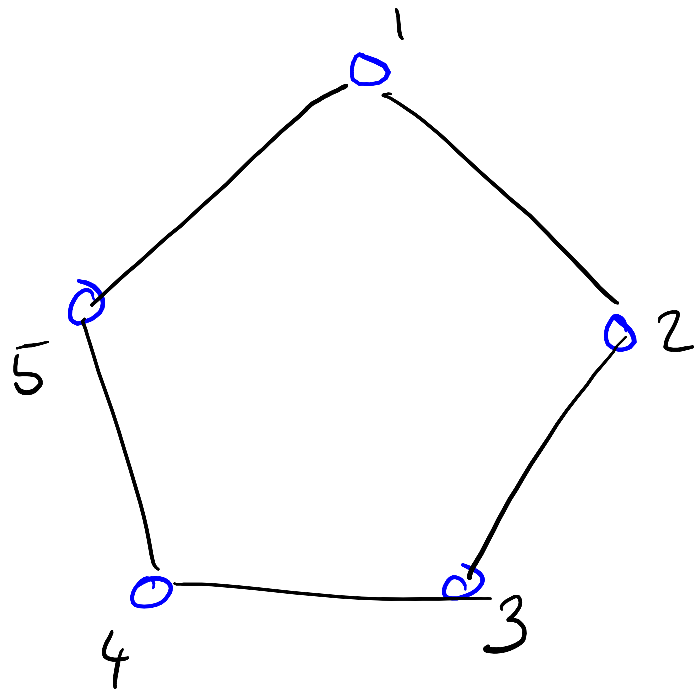


chord diagram

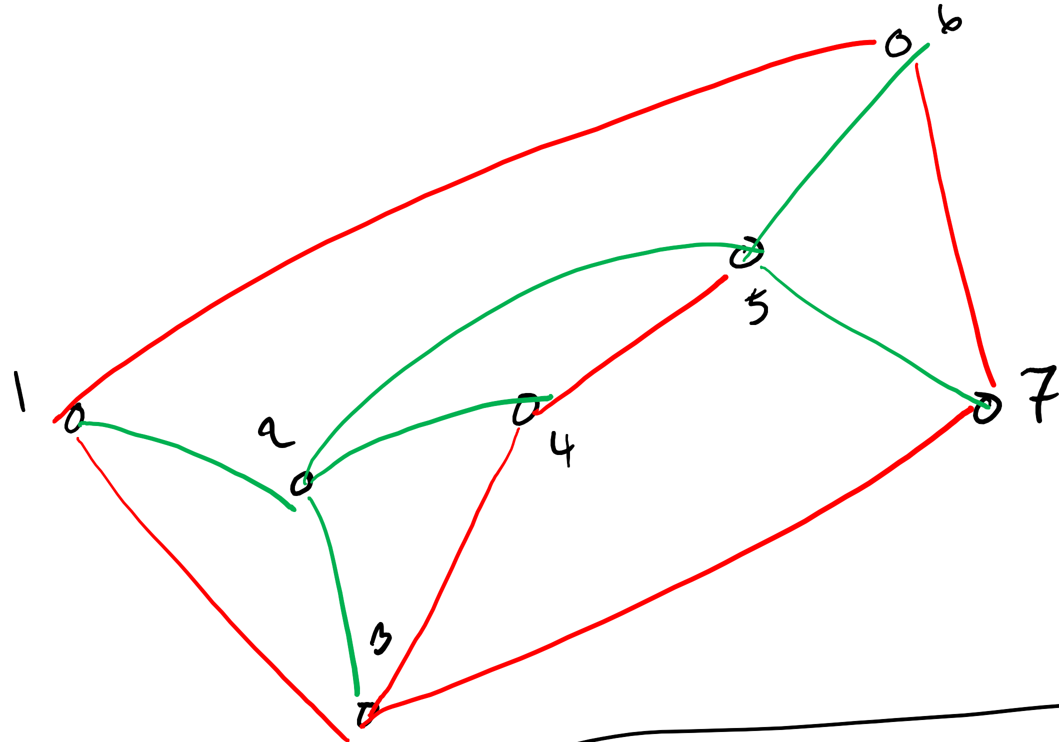
$\mathcal{C}$



circle graph

$G(\mathcal{C})$

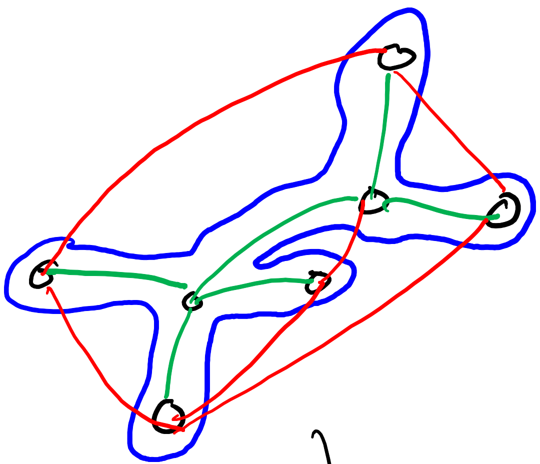
Let's start with something simple:



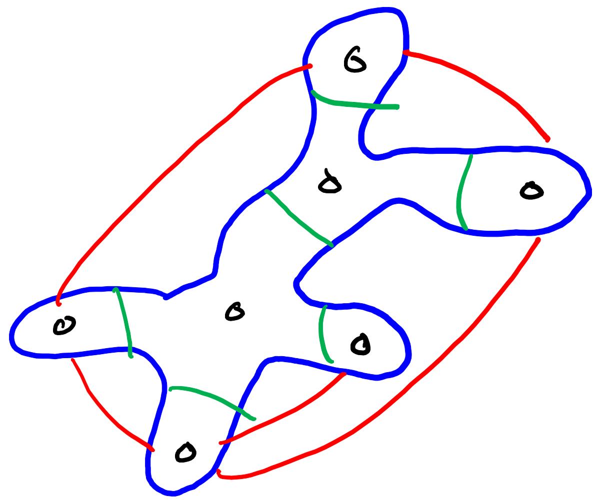
a planar graph

green edges form a spanning tree
red edges are everything else

Draw a circle around the spaning tree:

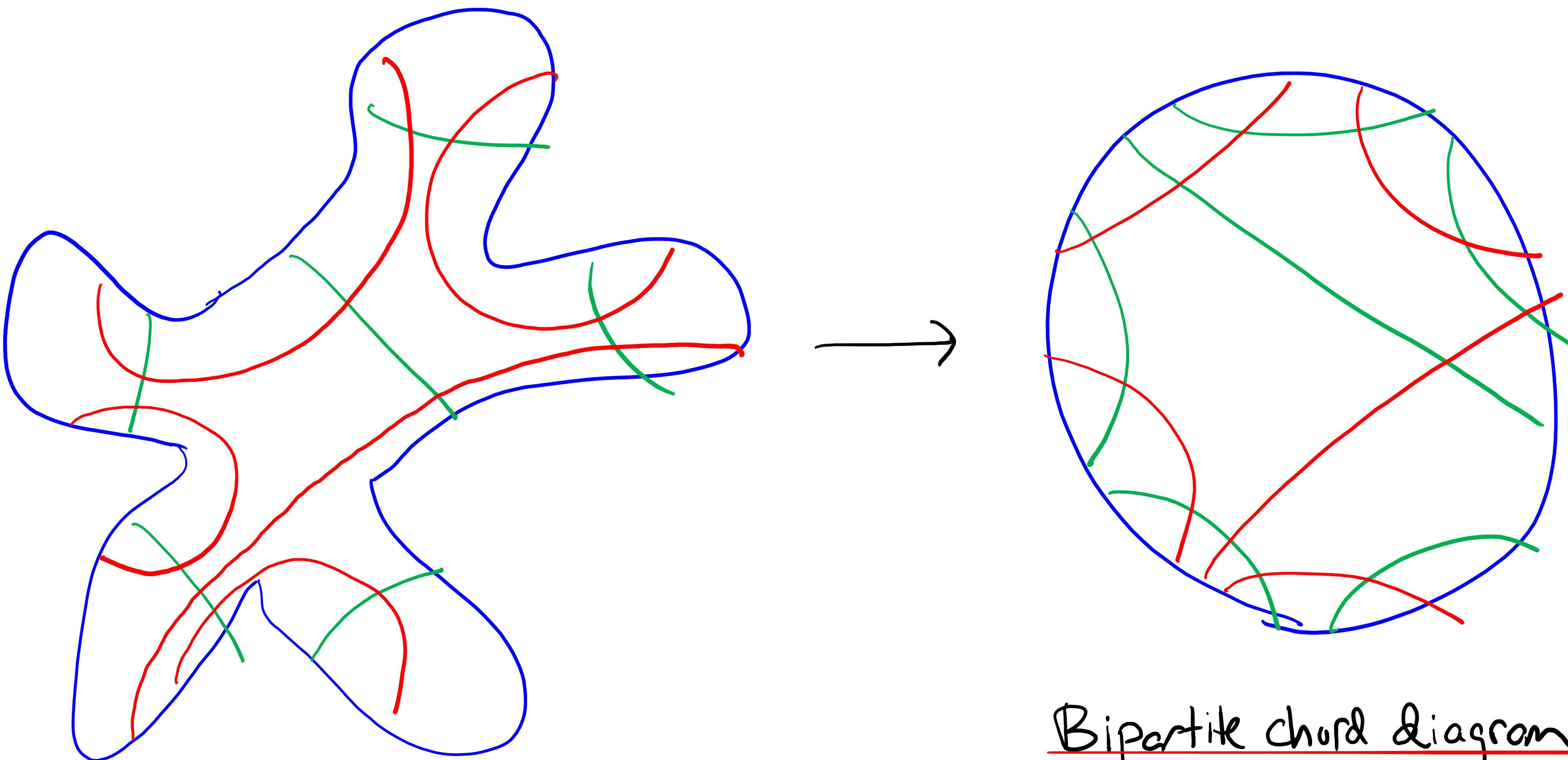


then replace the spaning tree with perpendicular chords:



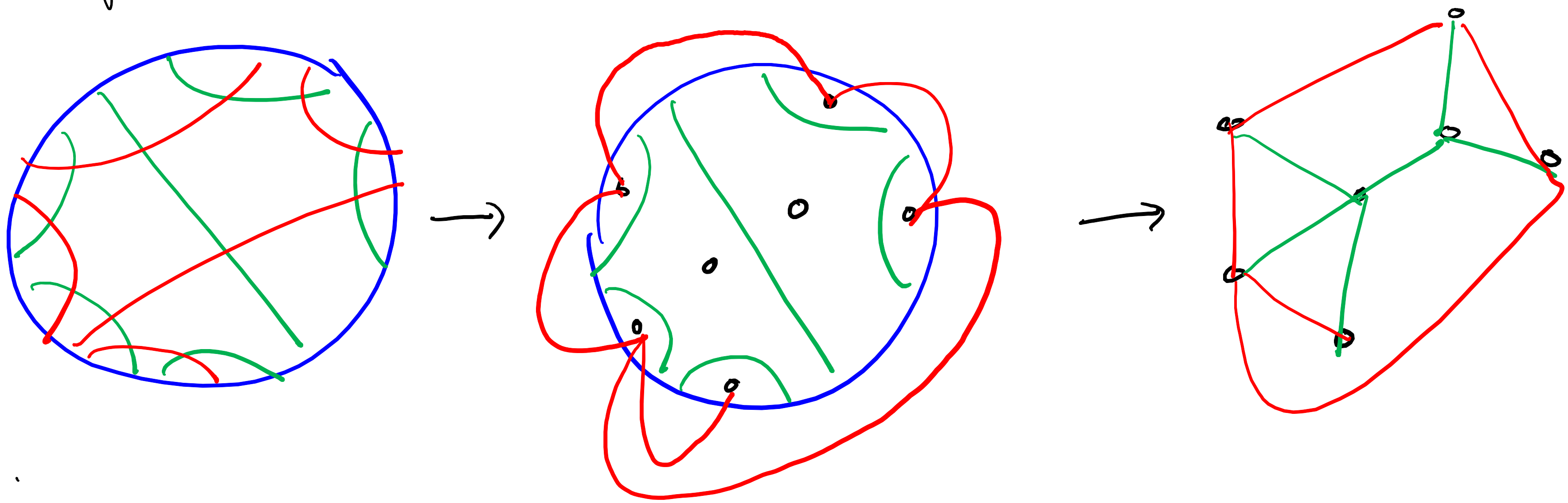
(also trim off the ends of the red edges)

Now flip the red chords into the circle, making sure they don't intersect:



Bipartite chord diagram

This process is also reversible:



Bipartite chord diagram ⟶ Planar Graph

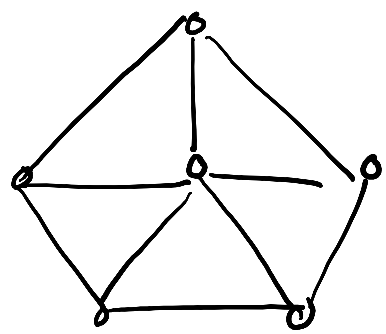
Theorem (De Frayessix)

There is a correspondence between bipartite circle graphs and planar graphs.

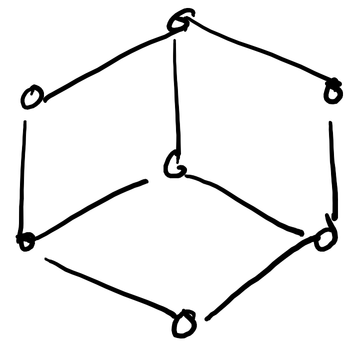
Hence characterizations of circle graphs lead to characterizations of planar graphs.

Bouchet's Theorem

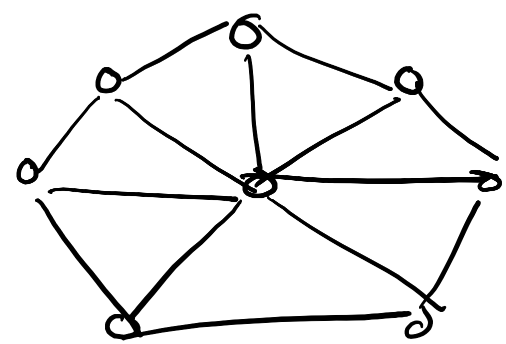
Theorem (Bouchet) The set of excluded vertex-minors for the class of circle graphs is:



W_5



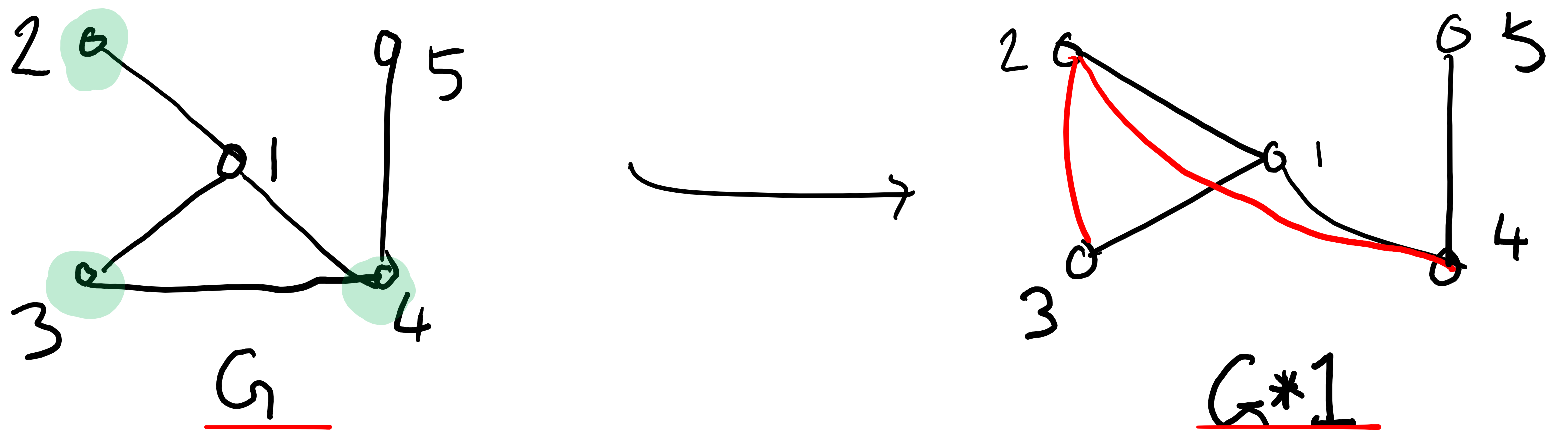
F_7



W_7

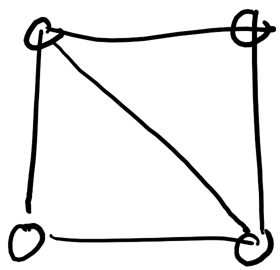
H is a vertex-minor of G if H can be obtained from G by:

- vertex deletion
- local complementation



Two graphs are **locally equivalent** if one can be obtained from the other by local complementations.

Ex: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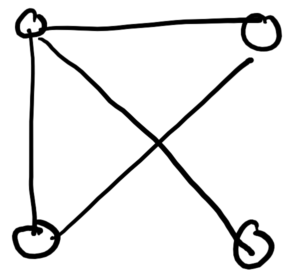
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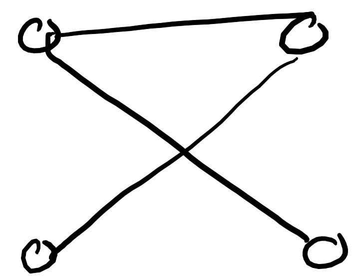


are **locally equivalent**.

Caution! Unlike edge contraction/deletion, vertex deletion and local complementation **do not commute!**

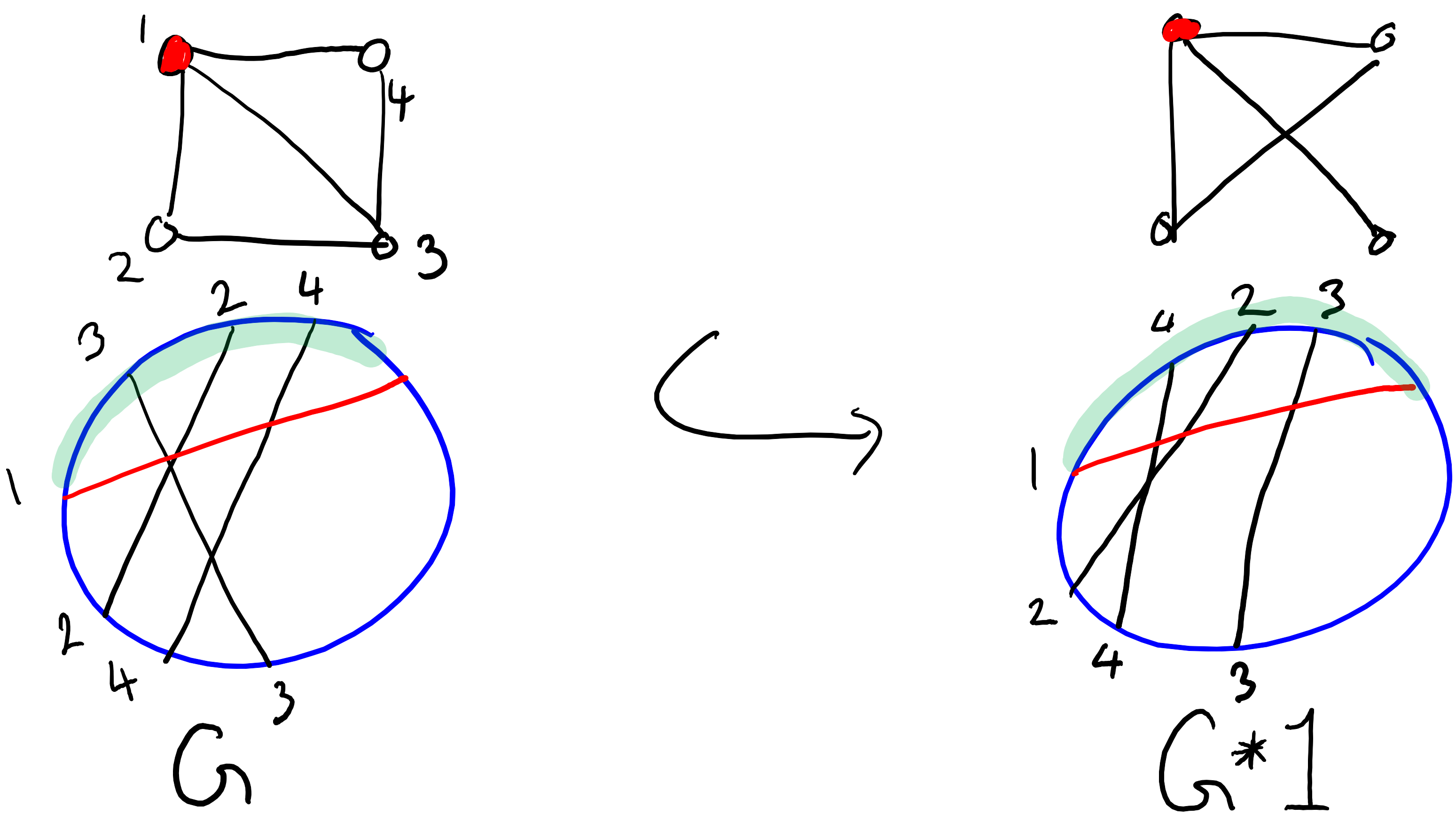
Lemma (Bouchet) Up to local equivalence, there are three ways to remove a vertex $v \in V(G)$:

1) $G \setminus v$ 2) $G * v \setminus v$ 3) $G * u * v * u \setminus v$

for any neighbour w of v in G.

NOTE: The choice of neighbour does not matter.

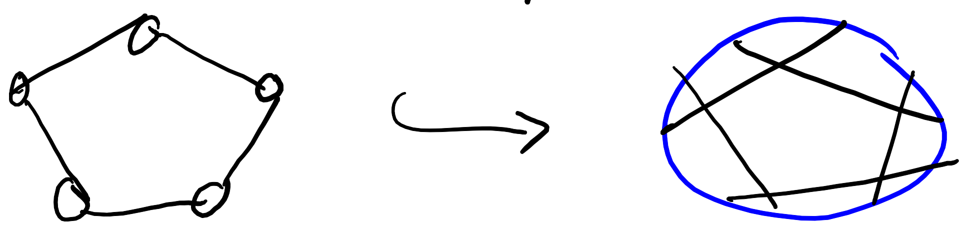
Naturally, circle graphs are closed under local complementation:



Back to **Bouchet's Theorem**: G is a circle graph $\iff$ no W_5, W_7, nor F_7 vertex-minor:

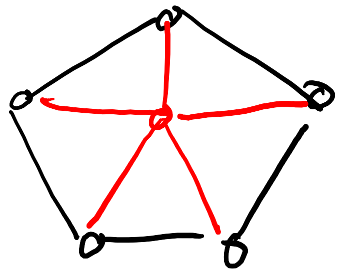
One direction is relatively easy:

* Cycles have unique chord diagrams

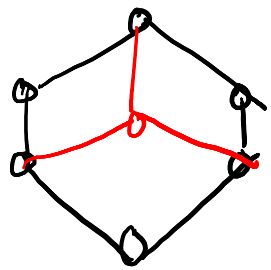


* Hence a vertex can't be adjacent to
    * Five or more vertices on a cycle
    * 3 or more non-adjacent vertices on a cycle.

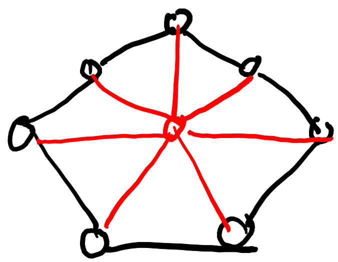
This rules out:



- 5 neighbours on an induced cycle.



- 3 non-adjacent neighbours on a cycle.



- 7 neighbours on a cycle.

Plan:

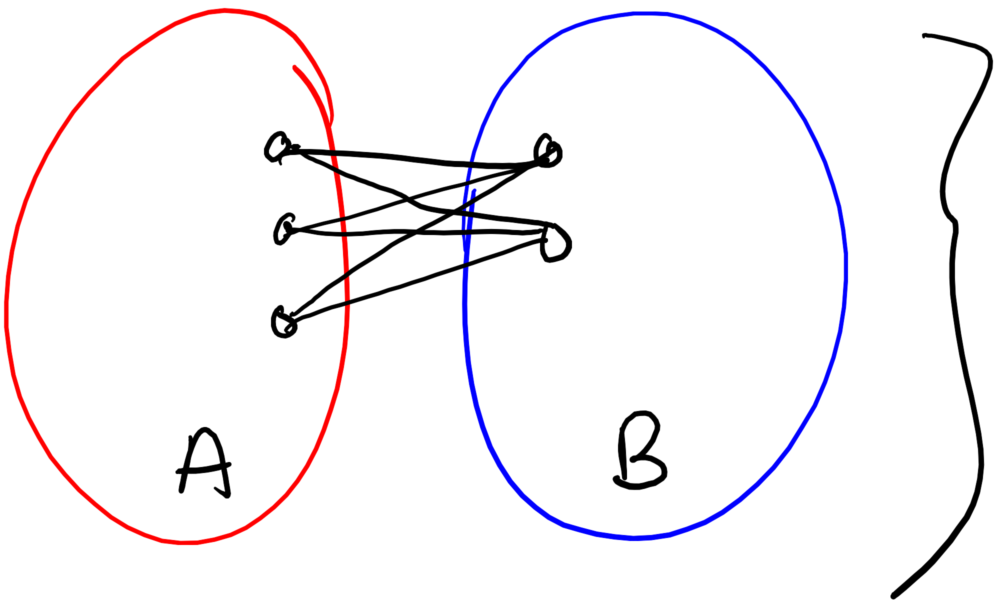
1. Introduce some connectivity machinery (analogue of 3-connectivity for planar graphs)

2. Develop a nice representation of circle graphs and extensions of circle graphs.

3. Characterize when extended circle graphs are not circle graphs.

Connectivity

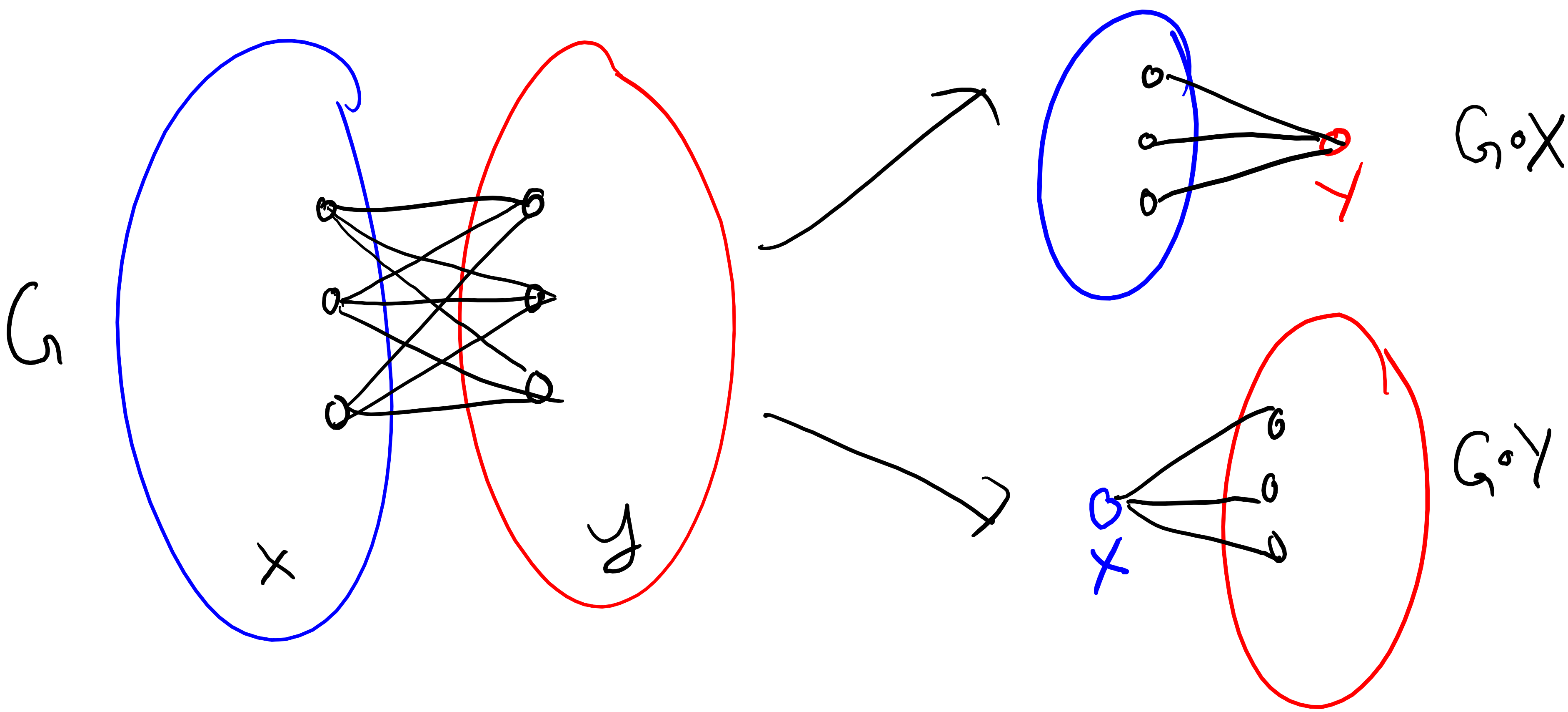
A split of a graph G is a bipartition (A, B) of $V(G)$ such that $G[A,B]$ is a complete bipartite graph, and $|A| \geq 2$, $|B| \geq 2$.



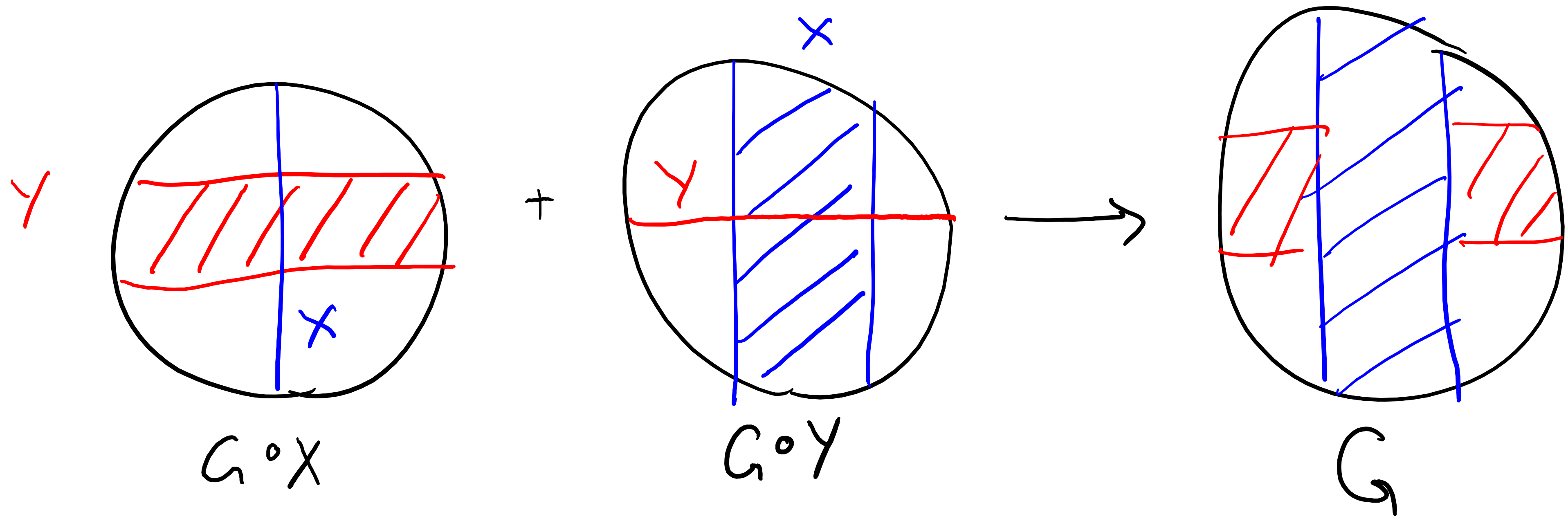
split.

A graph G is prime if it has no splits.

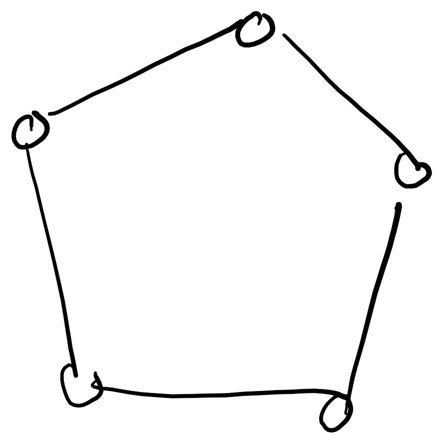
Lemma (various) Let G be a graph. If G has a split (X, Y), then G is a circle graph if and only if $G \circ X$ and $G \circ Y$ are:



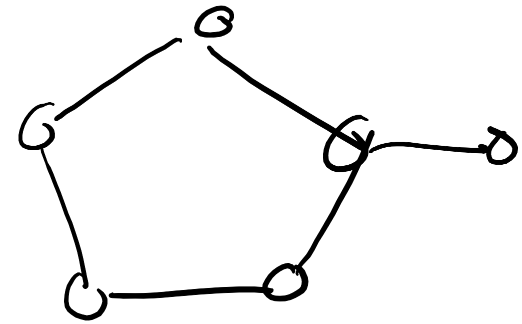
Proof $G \circ X$ and $G \circ Y$ are induced subgraphs of G.
Conversely,



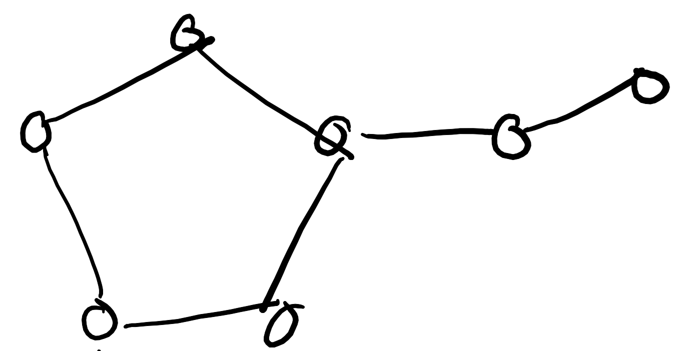
A graph G is internally prime if if all of its (A,B) splits have one of $|A| \leq 2$ or $|B| \leq 2$.



prime
internally prime



not prime
internally prime



not prime
not internally prime

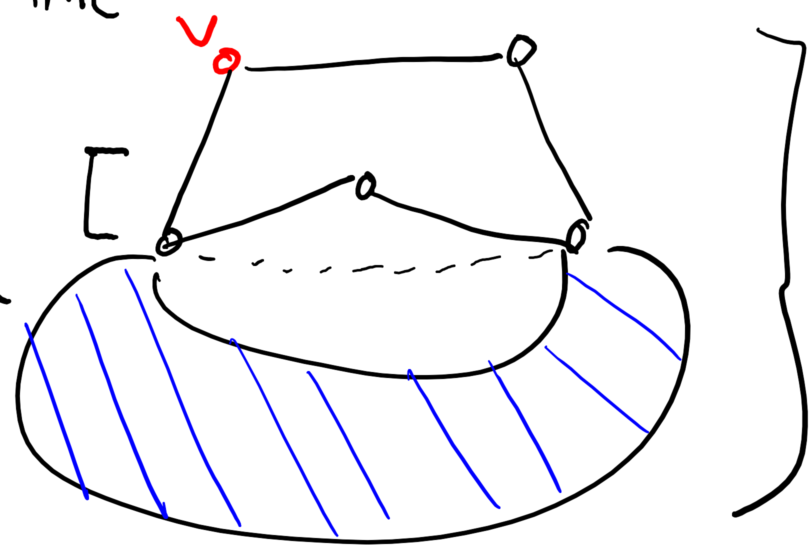
Lemma A (GL) If G is a prime graph, $v \in V(G)$, w a neighbour, then two of

- $G \setminus v$
- $G * v \setminus v$
- $G * v * w * v \setminus v$

are internally prime.

Lemma B (GL) If G is a prime graph, $v \in V(G)$, w a neighbour, then either

- one of $G \setminus v$, $G * v \setminus v$, or $G * v * w * v \setminus v$ is prime
-

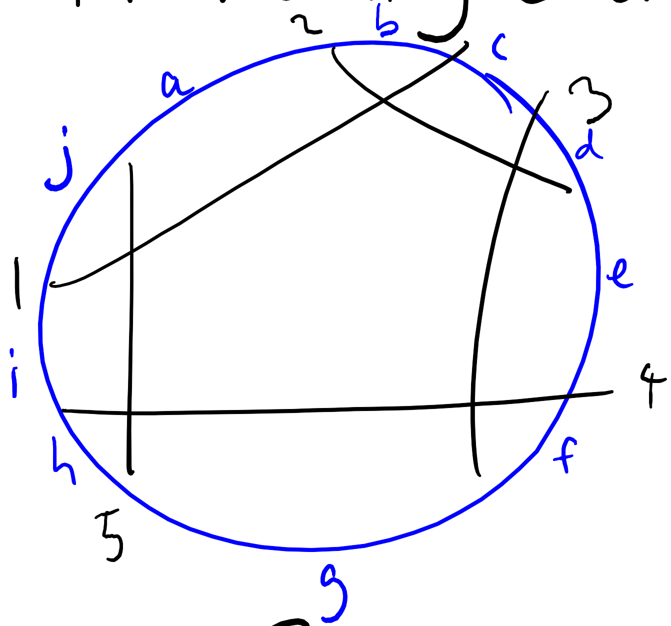


these vertices can be removed 2 ways preserving primality.

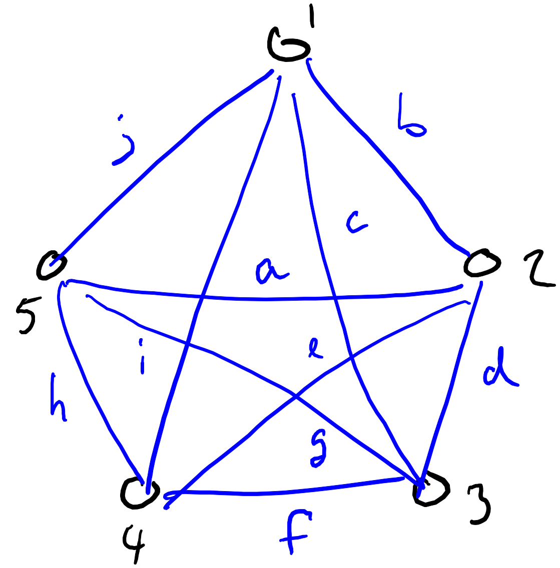
G (up to local equivalence)

Representations

From every chord diagram



$\mathcal{C}$

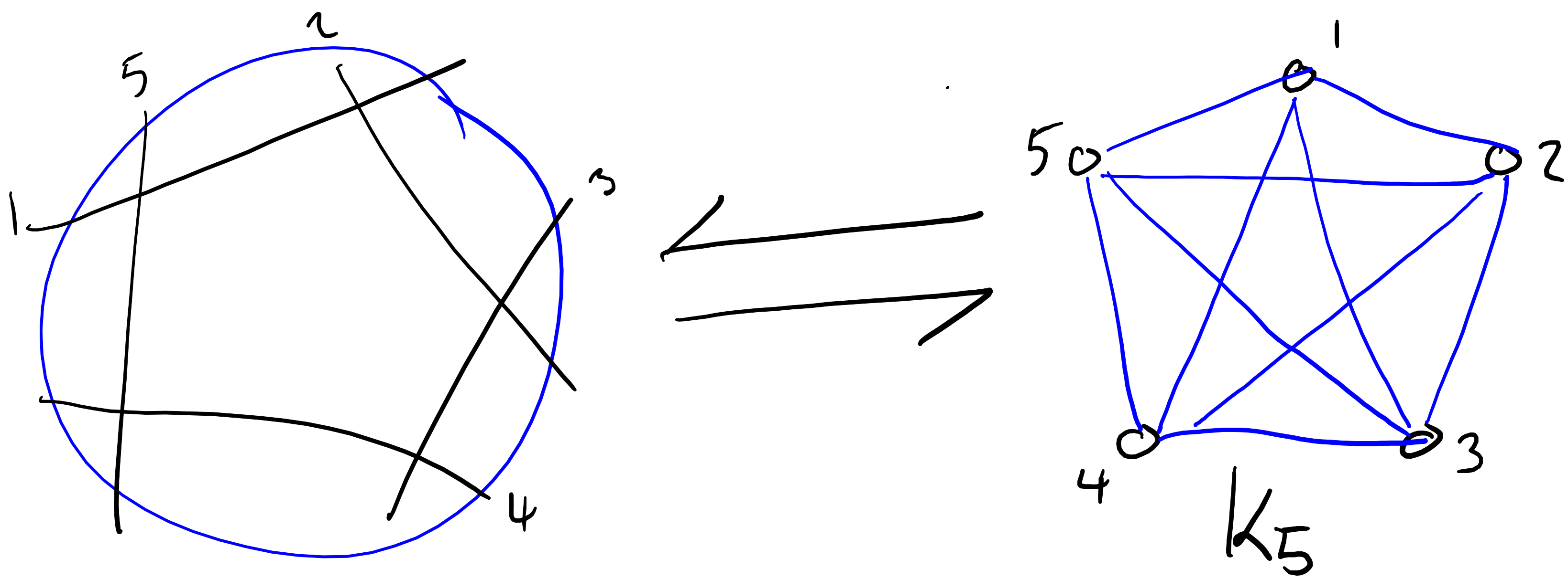


we can obtain an associated 4-regular graph R by treating chords as vertices and arcs as edges.

R is a <u>tourgraph</u> of $\mathcal{C}$.

this is a handy representation, as ....

* an Eulerian tour of R encodes a circle graph
(e.g the tour 1521324354 of K_5 encodes C_5)

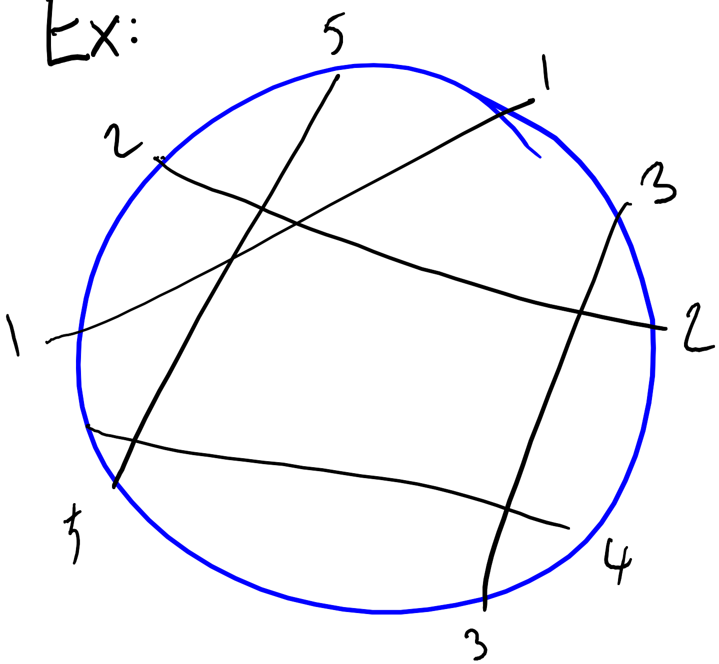


⊕

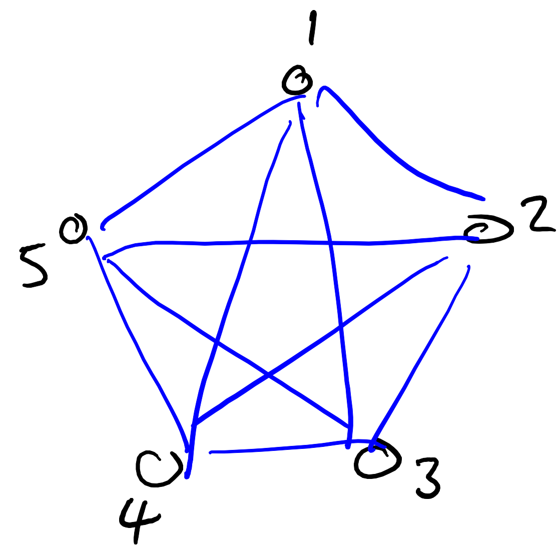
1, 5, 2, 1, 3, 2, 4, 3, 5, 4

* two different tours of R encode locally equivalent graphs — as reversing the order in which you traverse vertices does not affect the tour graph in any way.

Ex:

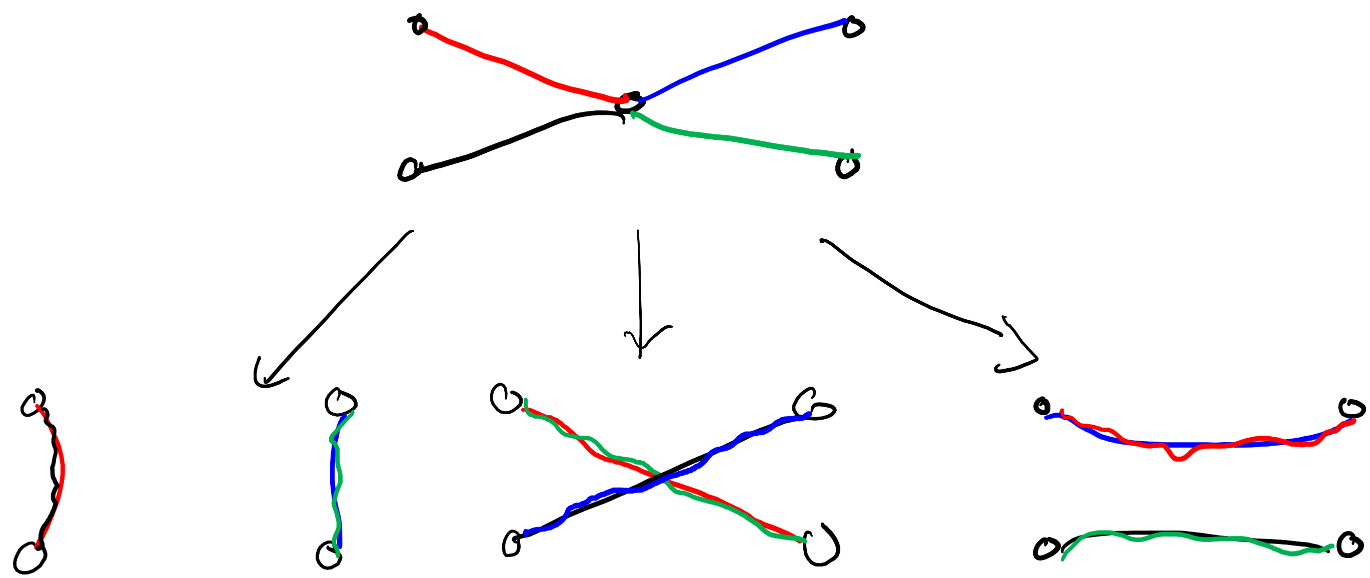


still has tour graph



<u>Lemma (Kotzig)</u> The set of tours of a 4-R-G R encode a local equivalence class of circle graphs

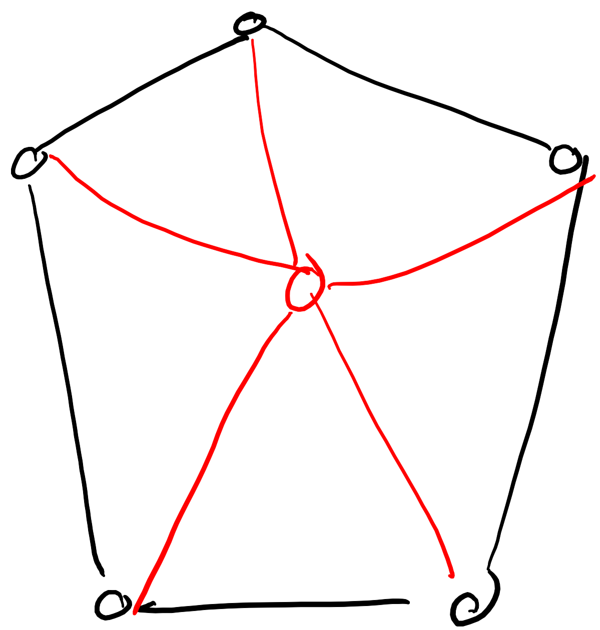
Moreover, the three ways to split a vertex in R



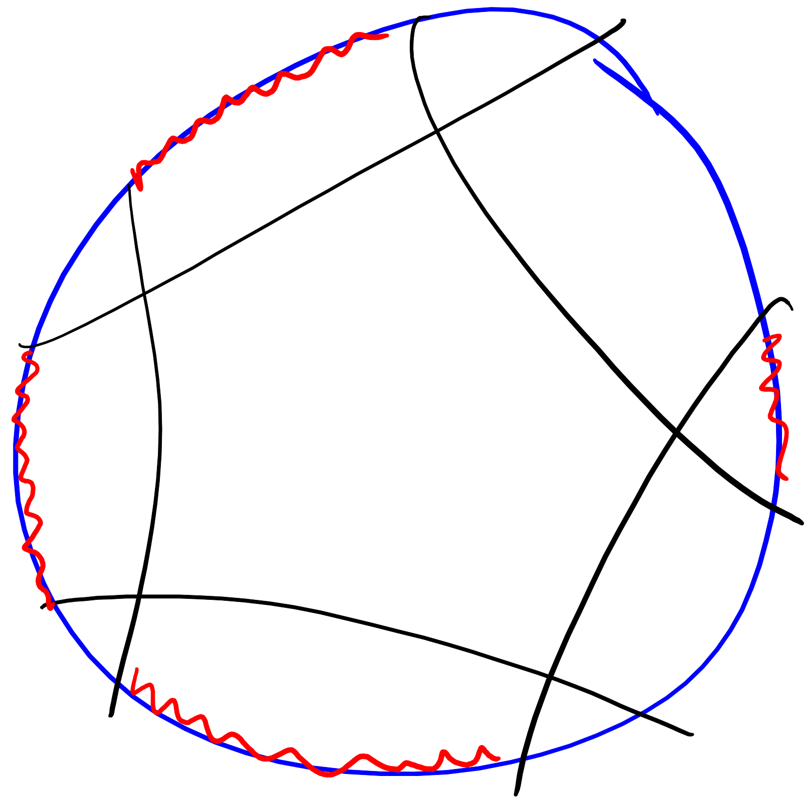
correspond to the three ways to remove a vertex in G (up to local equivalence).

Let's switch gears for a minute.

Q: How do we represent (nicely) extensions of circle graphs?



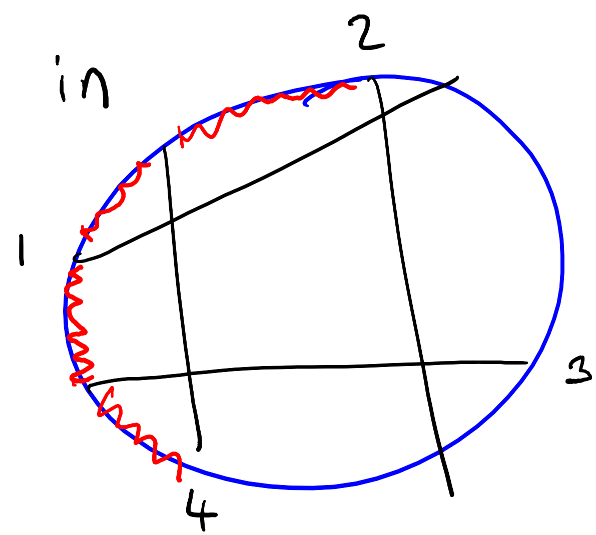
A: Hyperchords



A hyperchord Σ of a chord diagram $\mathcal{C}$ is an even set of arcs of Σ.

We say Σ crosses a chord v if v splits Σ into two odd parts.

E.g in

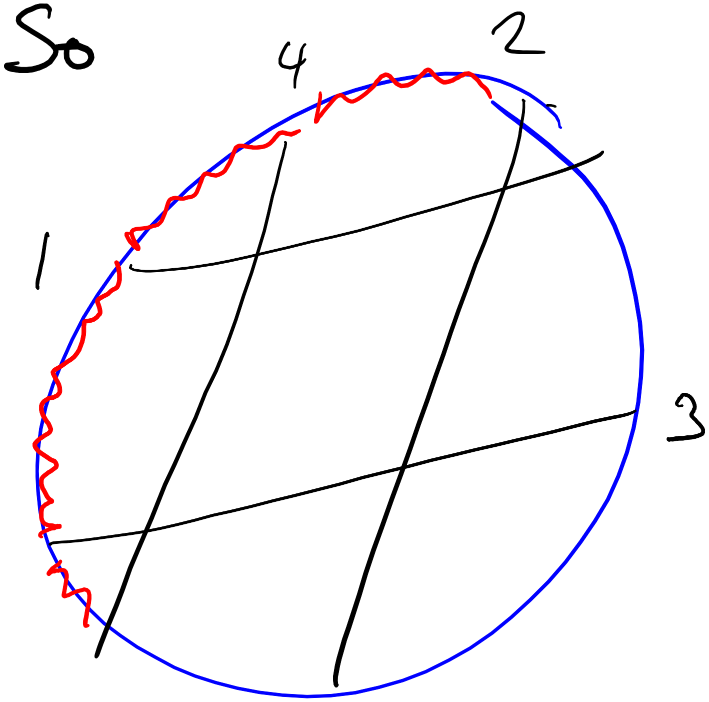


3 and 4 split Σ into 2 odd parts.

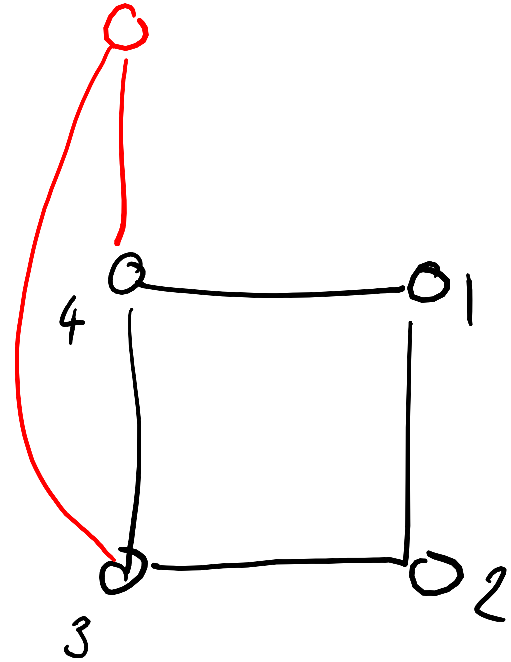
1 and 2 do not.

An arc is odd if it is in Σ.

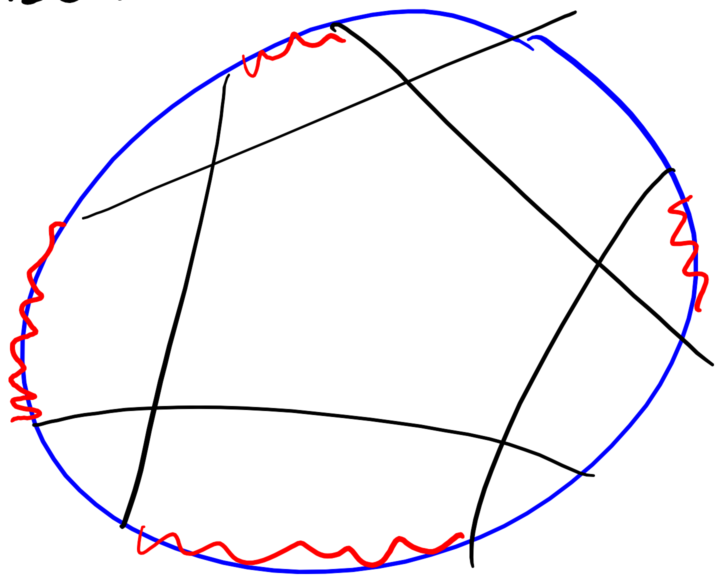
So



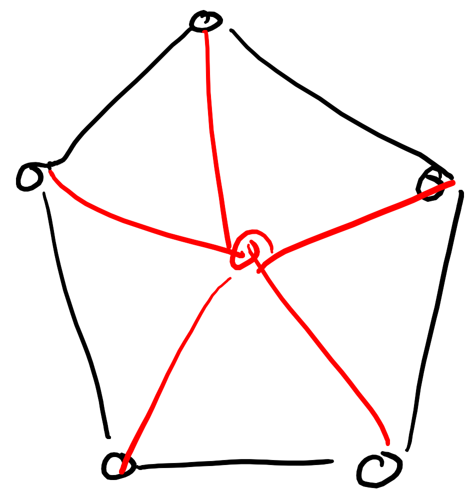
represents



Also:



represents



W_5

Delete chord – merge incident arcs preserving parity.

$$\left.\begin{array}{l}\text{Two odd arcs}\\ \text{Two even arcs}\end{array}\right\} \longrightarrow \text{1 even arc.}$$

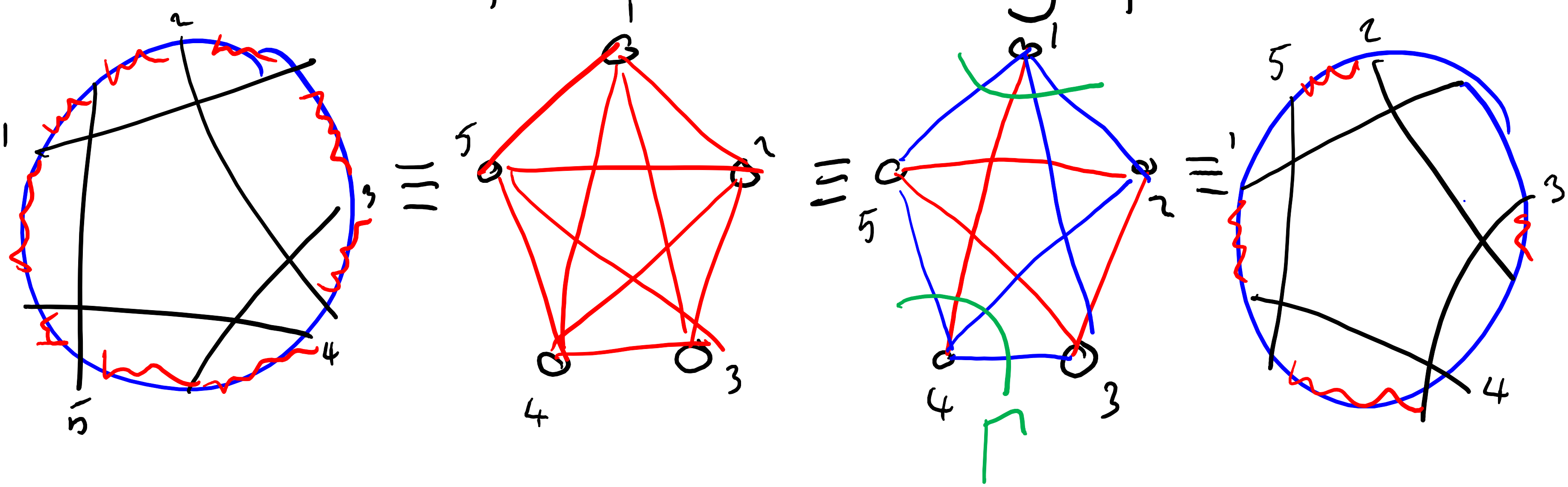
1 of each $\longrightarrow$ one odd arc.

Lemma R (Bouchet) Single-vertex extensions of chord diagrams $\mathcal{C}$ will always have a hyperchord representation $(\mathcal{C}, \Sigma)$

Proof Idea: Parity constraints can be encoded as a liner system of equations.

Now a set Σ of arcs of $\mathcal{C}$ is simply a set of edges of R.

Let Γ be a cut of R. Note that (C, Σ) and $(C, \Sigma \Delta \Gamma)$ represent the same graph.



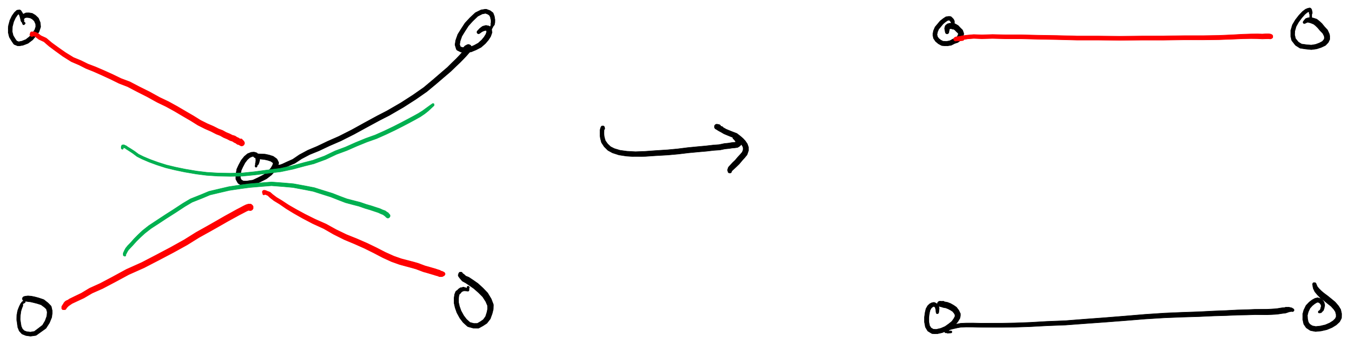
Moreover:

Lemma (Bouchet) If $(\mathcal{C}, \Sigma_1)$ and $(\mathcal{C}, \Sigma_2)$ represent the same graph, then $\Sigma_1 \Delta \Sigma_2$ is a cut of $R(\mathcal{C})$.

Proof idea: count solutions to Lemma R

Hence the set of solutions of Lemma R correspond to an equivalence class of signed graphs on R.

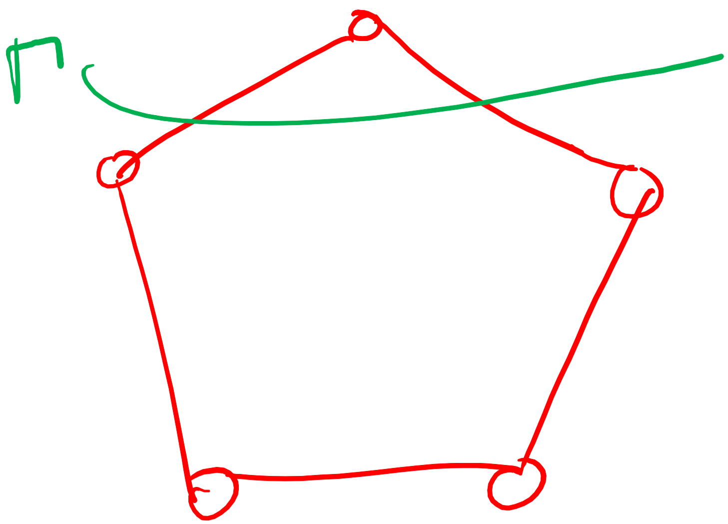
Note: splitting a vertex lifts up to signed graphs.
Merge identified edges preserving parity:



Now, if a hyperchord has two arcs, it really is just a chord.

Q: When does there not exist a hyperchord with only 2 arcs?

One problem is that there are two many cycles of R with an odd number of arcs of Σ.



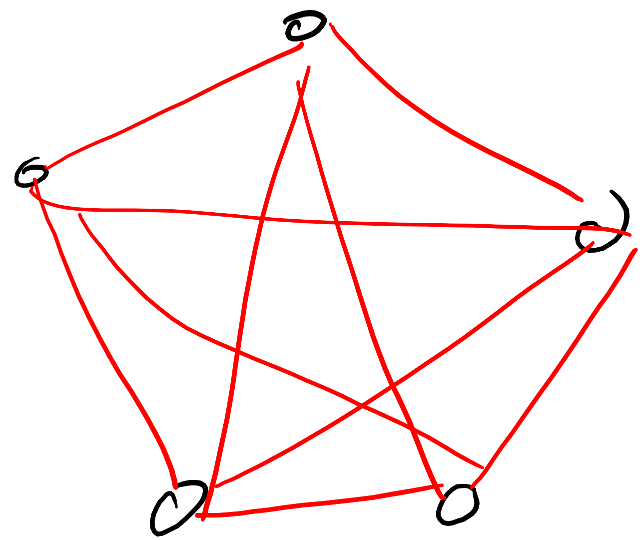
odd cycle C

as cuts intersect cycles in even parity, such odd cycles are invariant under resigning by a cut.

So every edge-disjoint odd circuit contributes one arc to Σ.

Or R could contain a special subgraph that will always intersect Σ in at least ~~3/4~~ arcs.

Ex: odd-K_5 = $(K_5, E(K_5))$



Will always intersect Σ in ≥ 4 arcs.

Note: odd-K_5 represents W_5.

Note that the blocking conditions never use ρ, the chord diagram, only R and Σ; we only care about cuts Γ of R.

So the real question we're trying to answer is:

Q: When is a 4-regular even signed graph (R, Σ) with even signature Σ not resignable to two odd edges?

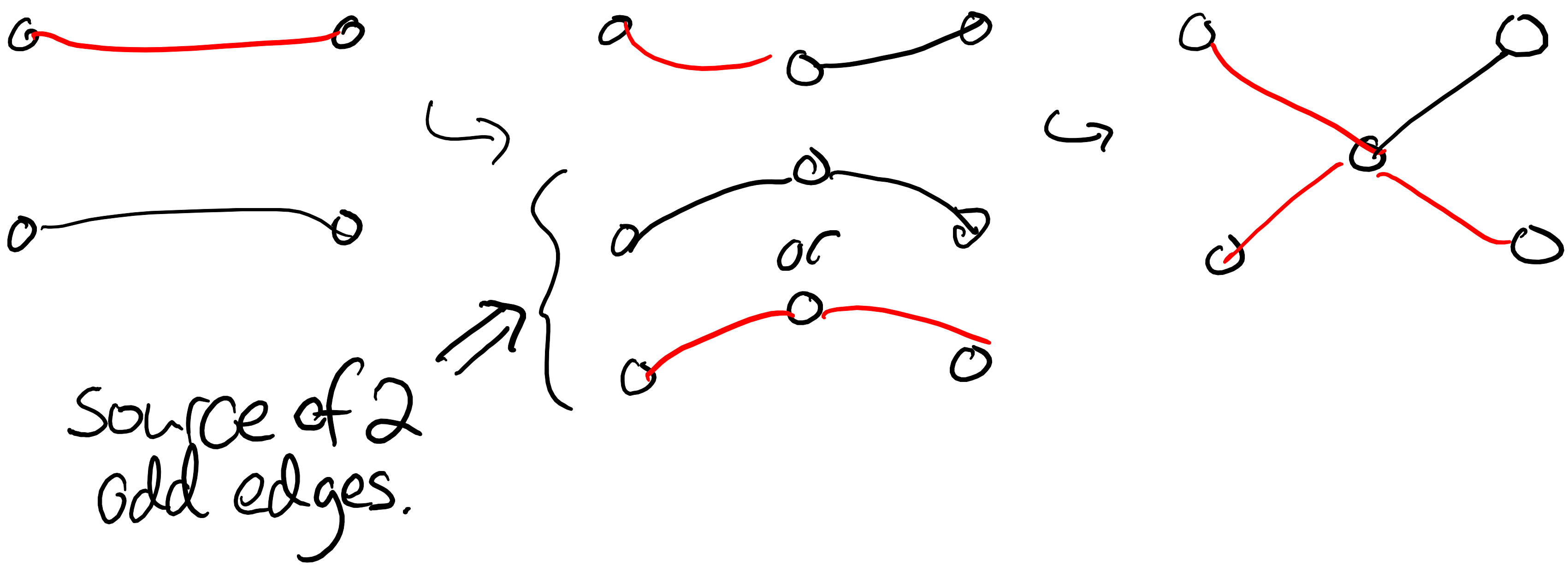
Lemma (G,L) Let (R,Σ) be a 4-regular even signed graph. Then either:

1) There is a cut Γ of R such that $|\Sigma \Delta \Gamma| = 2$.

2) (R,Σ) contains odd-K_5 as an immersion (obtained via a seqence of splits).

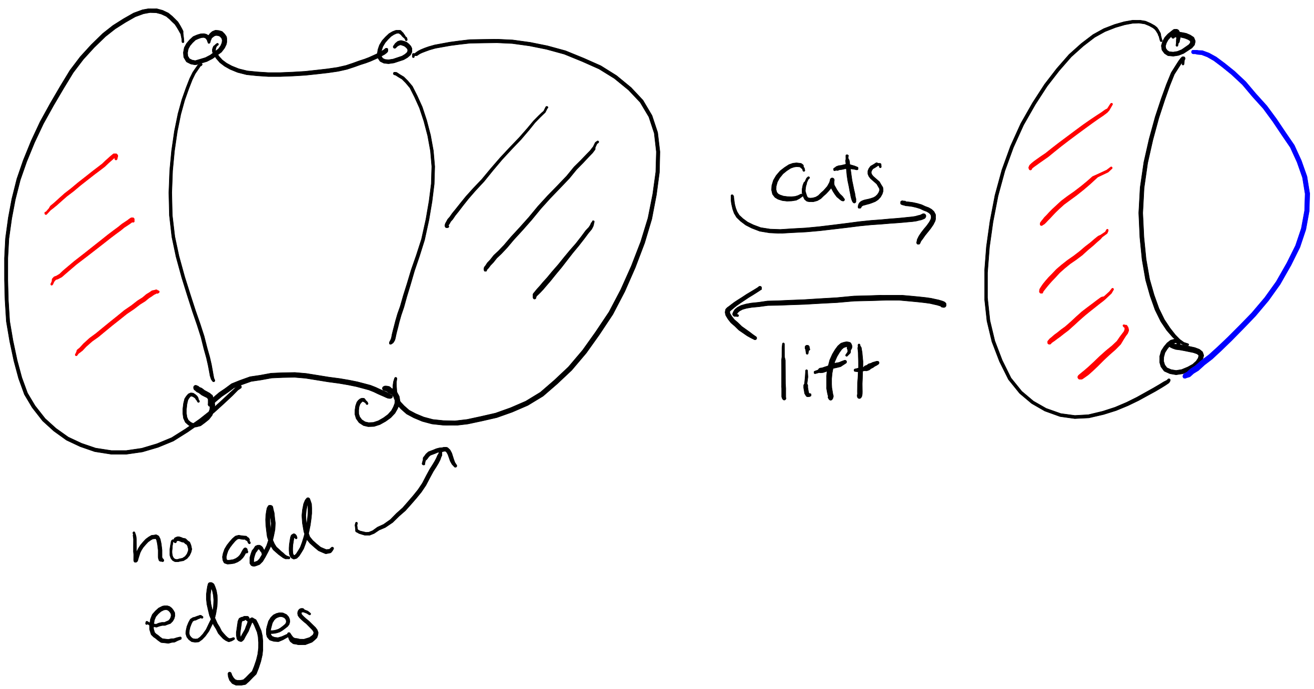
3) (R,Σ) has 3 edge disjoint odd circuits.

Note: We only need to prove the lemma for $|\Sigma| \leq 4$, as lifting a signature through a split adds at most 2 edges to it

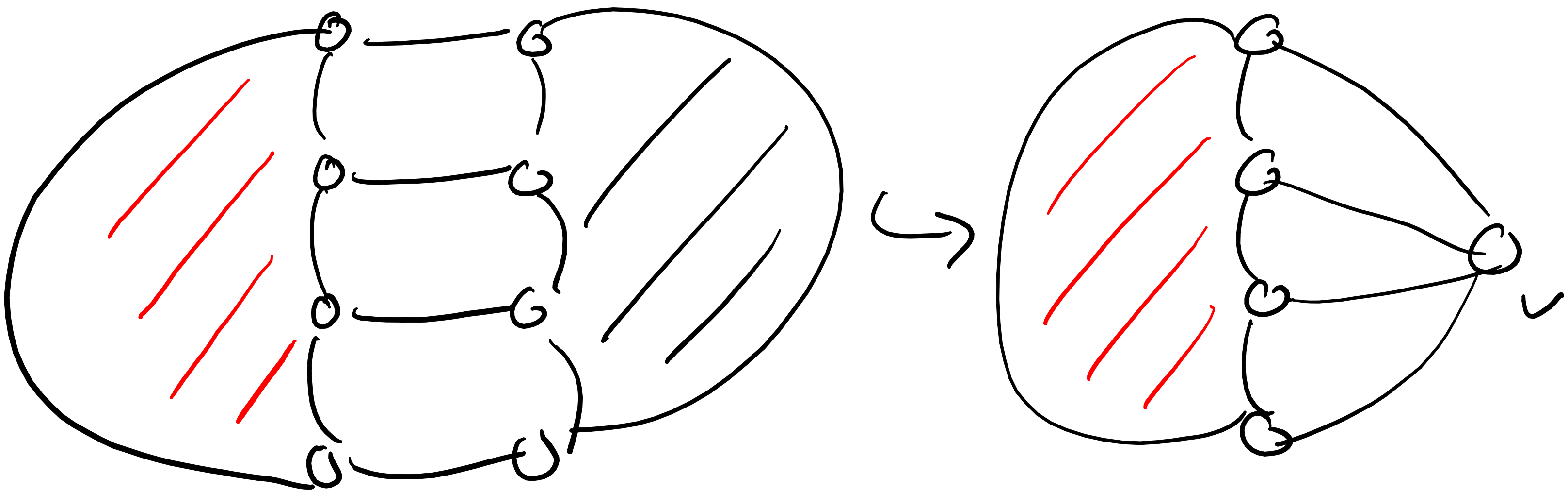


Unsplitting is equivalent to subdividing two edges and identifying the new vertices.

Reduction: R does not have a 2-edge-cut with one side balanced.

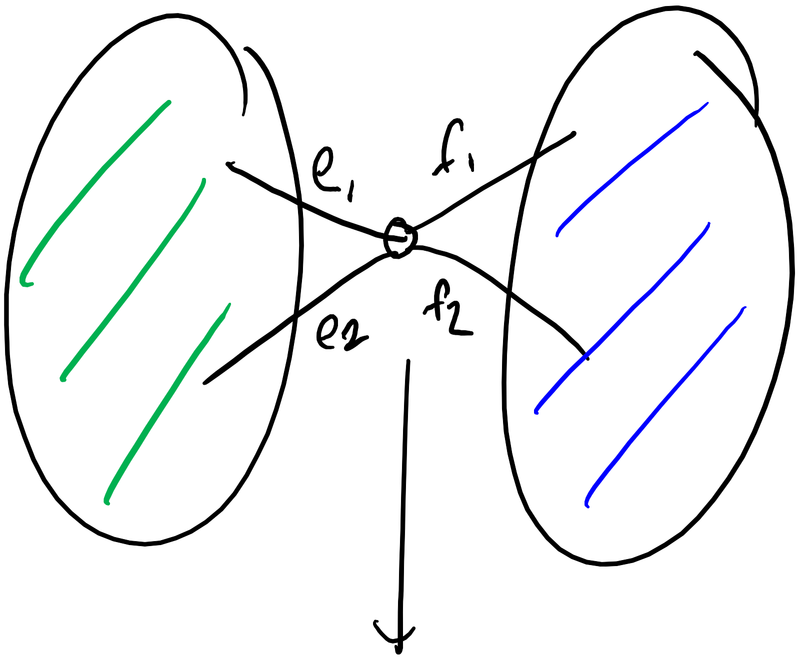


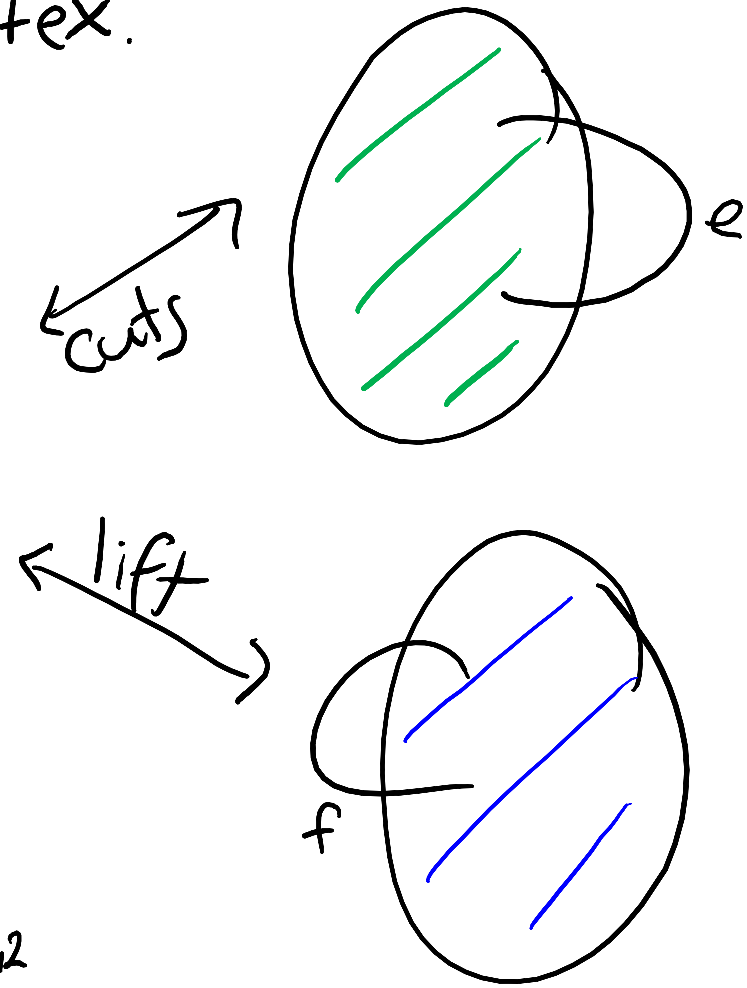
Reduction: no 4-edge-cut with one side balanced.



Cuts lift, and this immersion always exists since there is no 2-edge cut with one side balanced.

Reduction: no cut vertex*.





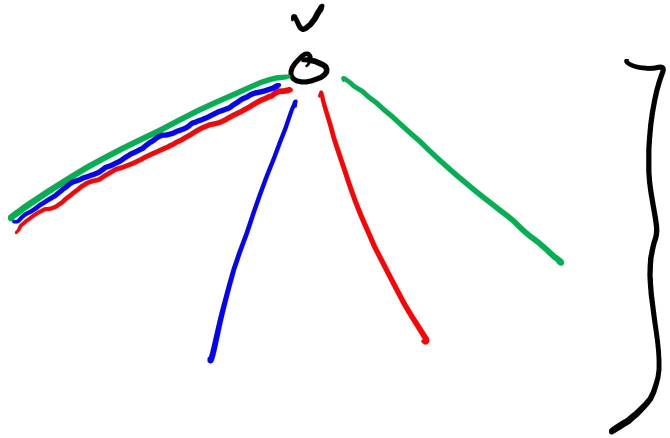
can assume if e/f is even then so is $e_{1,2}/f_{1,2}$ by resigning.

* need to weaken induction conditions but for simplicity we'll assume this just works.

Two cases:

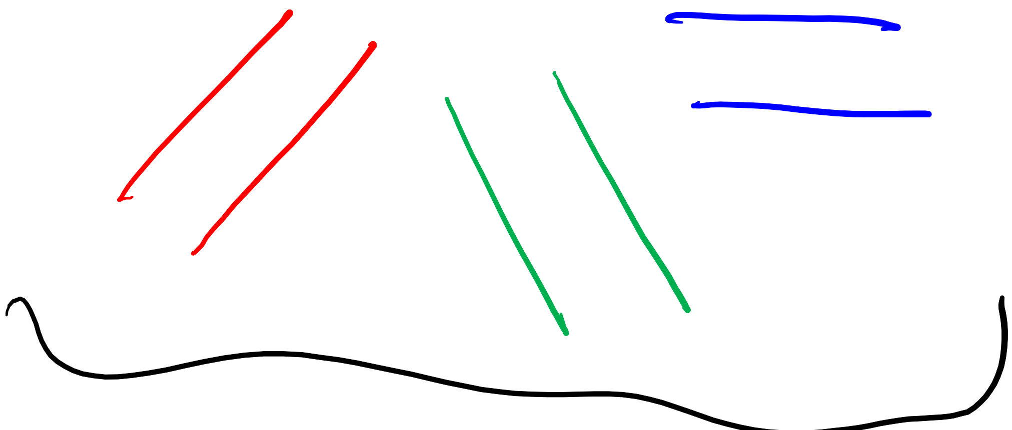
* Every vertex is in a parallel pair – easy.
  Every parallel pair is odd.

* Some vertex v is not in a parallel pair.



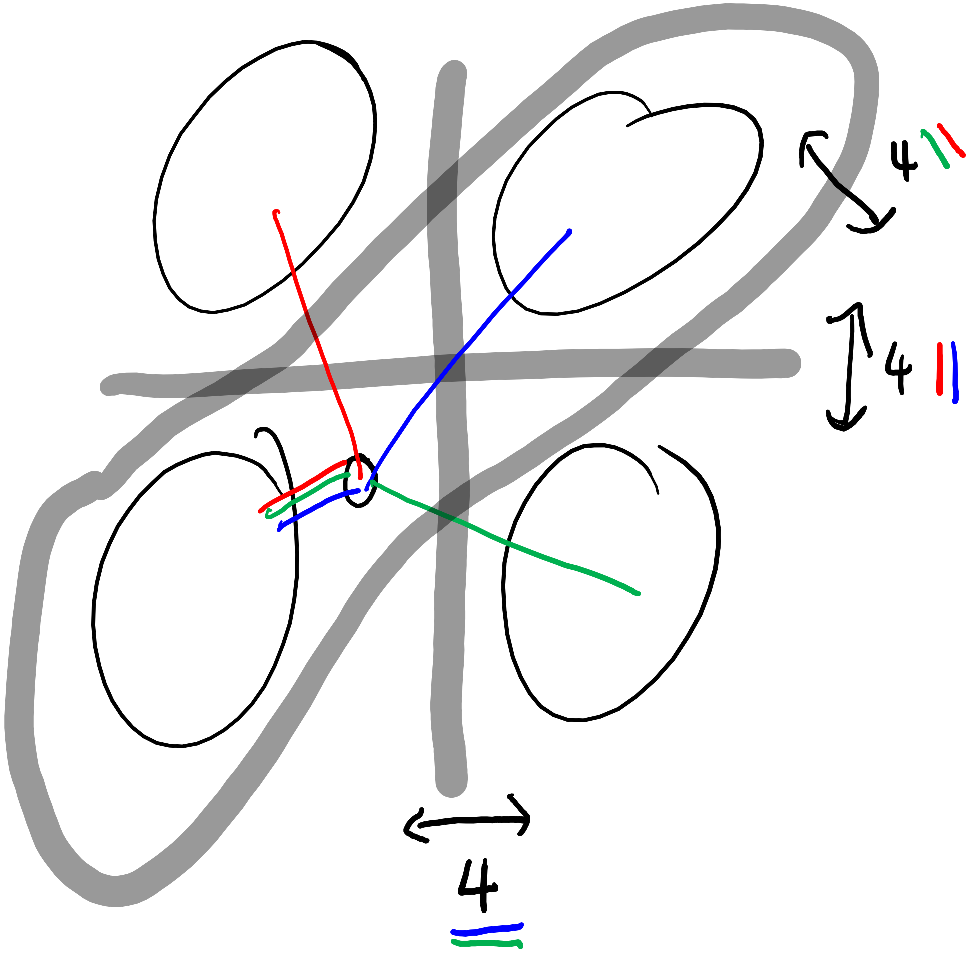
3 different ways to split, 3 different signatures.

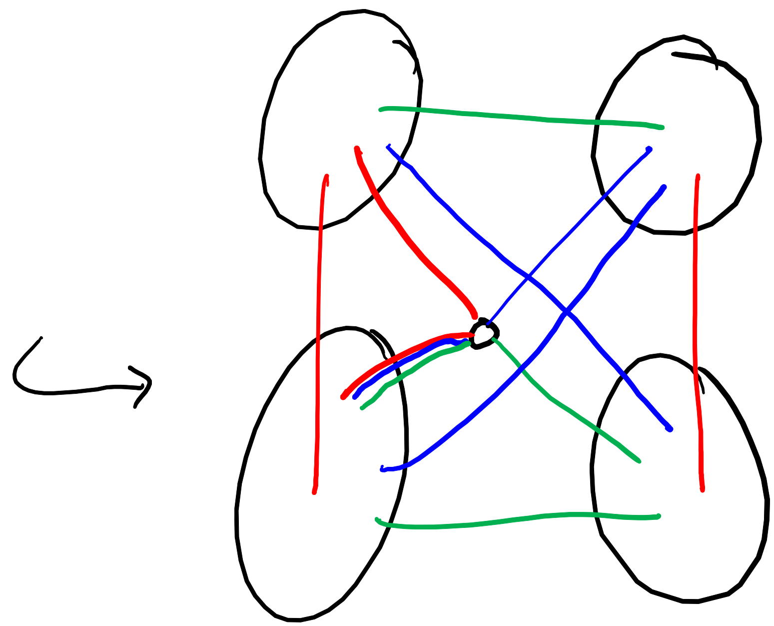
Note each signature has 2 edges on v as otherwise we'd apply induction and finish.



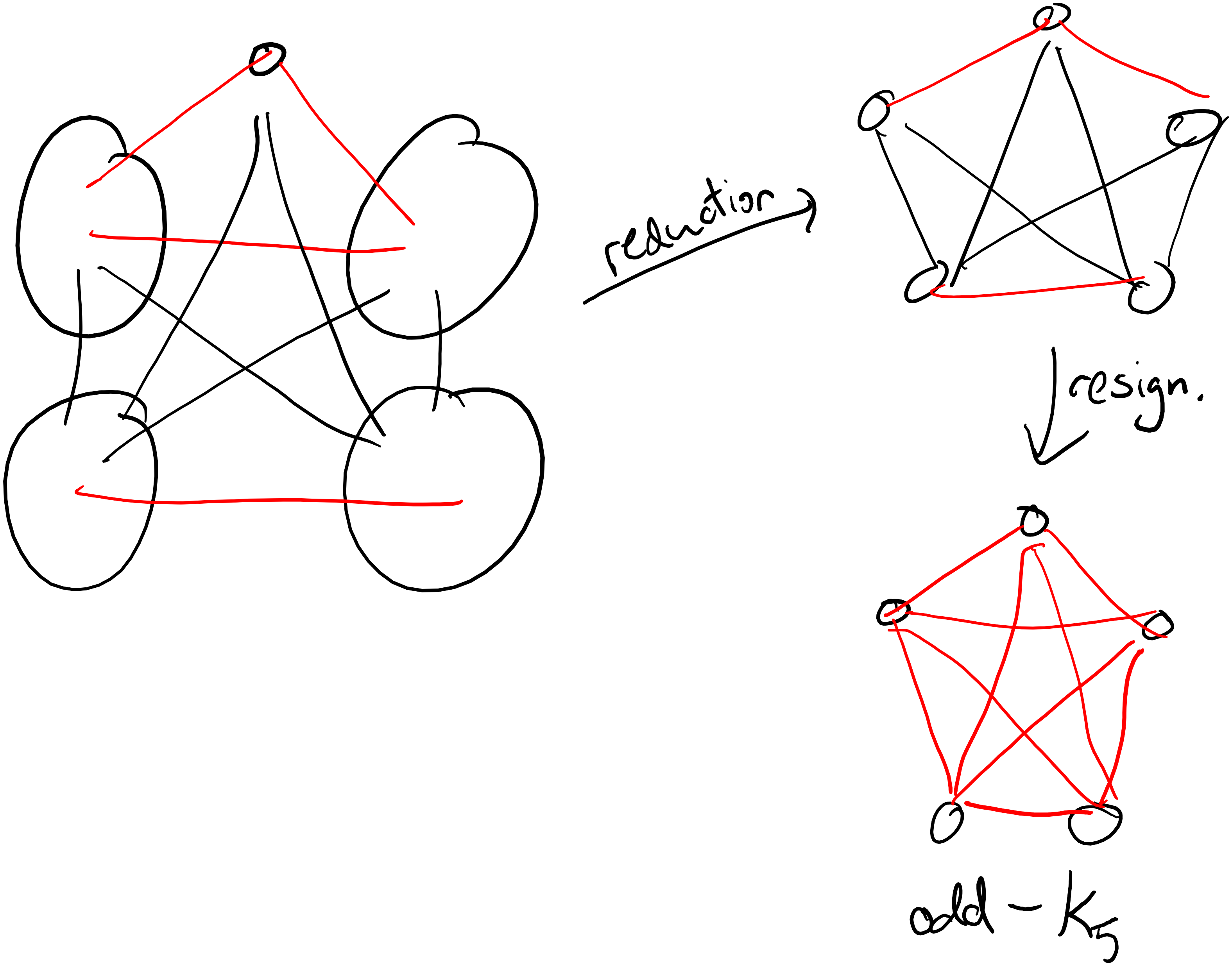
Also, each signature must have 2 edges and be disjoint relative to the rest of the graph as we'd run into connectivity issues.

Hence 3 signatures → 3 cuts of size 6 which split the graph into 4 components.





Now as no 4-E-C with one side balanced...


reduction
resign.
odd-K_5

So Lemma C tells us if (R,Σ) does not represent a circle graph then:

- (R,Σ) has an odd-K_5-immersion – good! G has a W_5-minor.

- (R,Σ) has 3 edge-disjoint odd circuits – need some more work.

First, however, a connectivity interlude...

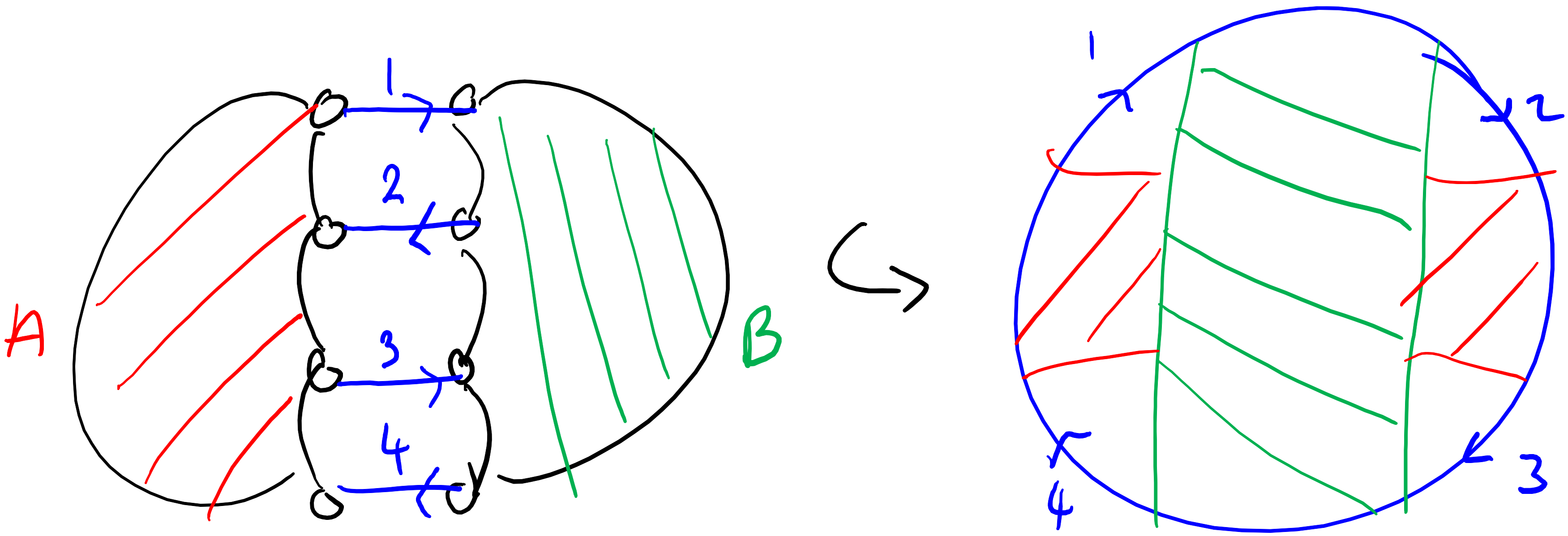
Connectivity Redux

R is <u>internally six-edge-connected</u> if $|\delta(X)| \geq 6$ when $|X| \geq 2$, $|V(G) \setminus X| \geq 2$

R is <u>weakly six-connected</u> if $|\delta(X)| \geq 4$ for all $X \subseteq V$ and $|\delta(X)| = 4 \iff |X| = 2$ or $|V(G) \setminus X| = 2$.

<u>Lemma</u> R is internally/weakly-six-connected iff G is prime/internally prime.

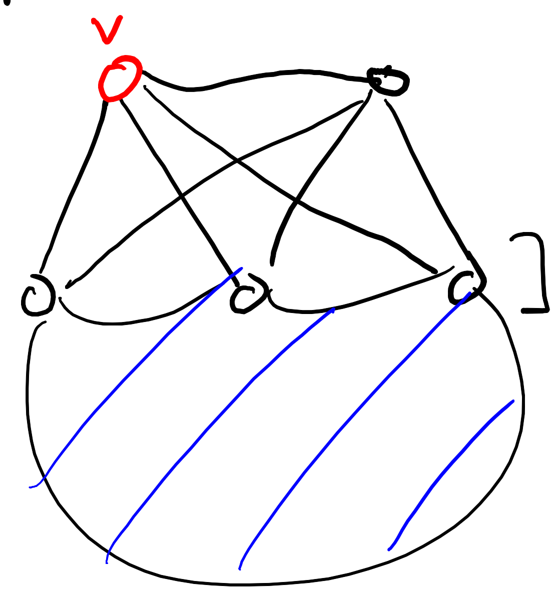
Sketch (Forwards)



So we get "R-versions" of Lemmas A and B.

Lemma A' Let R be an internally-6-connected 4-regular graph, $v \in V(R)$. Then there are two ways to split v preserving weak-6-connectivity

Lemma B' Let R be an I6C 4-R-G, $v \in V(R)$. Either there is a way to split v to preserve I6C, or:



] – these vertices can be split in 2 ways preserving I6C.

Excluded Representations

Have (from Lemma C)

- four regular internally-six-connected signed graph (R, Σ).
- Fixed set of 3 odd circuits C_1, C_2, C_3.
- no immersion preserves circuits/connectivity (minimality)

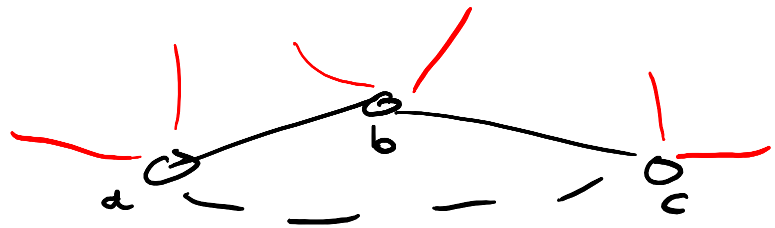
Claim: 4^{th} odd circuit comes for free.

Proof: Eulerian graph admits a cycle decomposition across C_1, C_2, C_3. Since there's an even # of odd edges, even # of odd cycles.

So we have a fixed set C_1, C_2, C_3, C_4 of odd circuits of R.

<u>Claim:</u> Every vertex v in R is incident to two of $C_1 \ldots C_4$.

Proof If one uncovered vertex then there are 3 uncovered vertices on a path due to parity.



Now by Lemma B' there is a way to remove one of a, b, c preserving I6C.

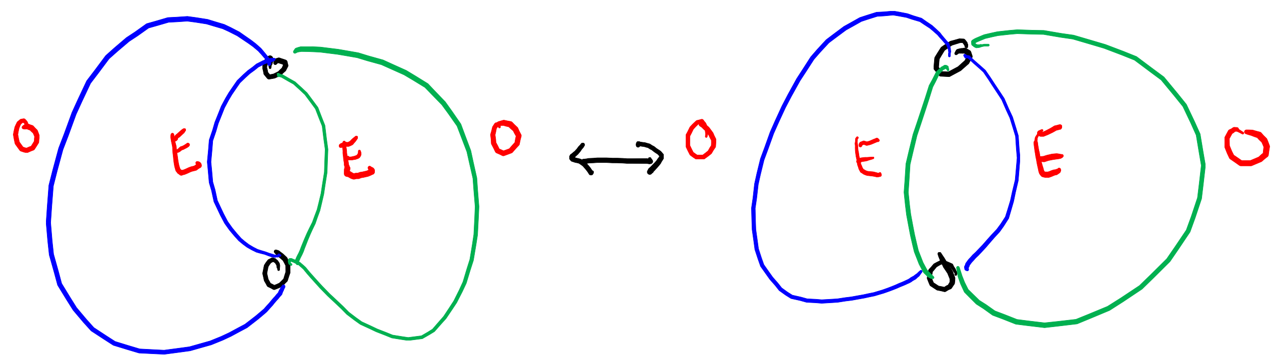
This destroys at most one odd circuit — a contradiction, as a 4RG signed graph with an even number of odd edges has an even number of odd circuits — 3 odd circuits $\rightarrow$ 4th for free.

$\blacksquare$

Claim: If two odd circuits meet twice, then there is a way to split preserving all odd circuits and weak 6-connectivity.

Proof: There are two ways to split apart a vertex while remaining weakly 6-connected.

There also two possible ways to reroute odd circuits that meet twice at one of their common vertices.

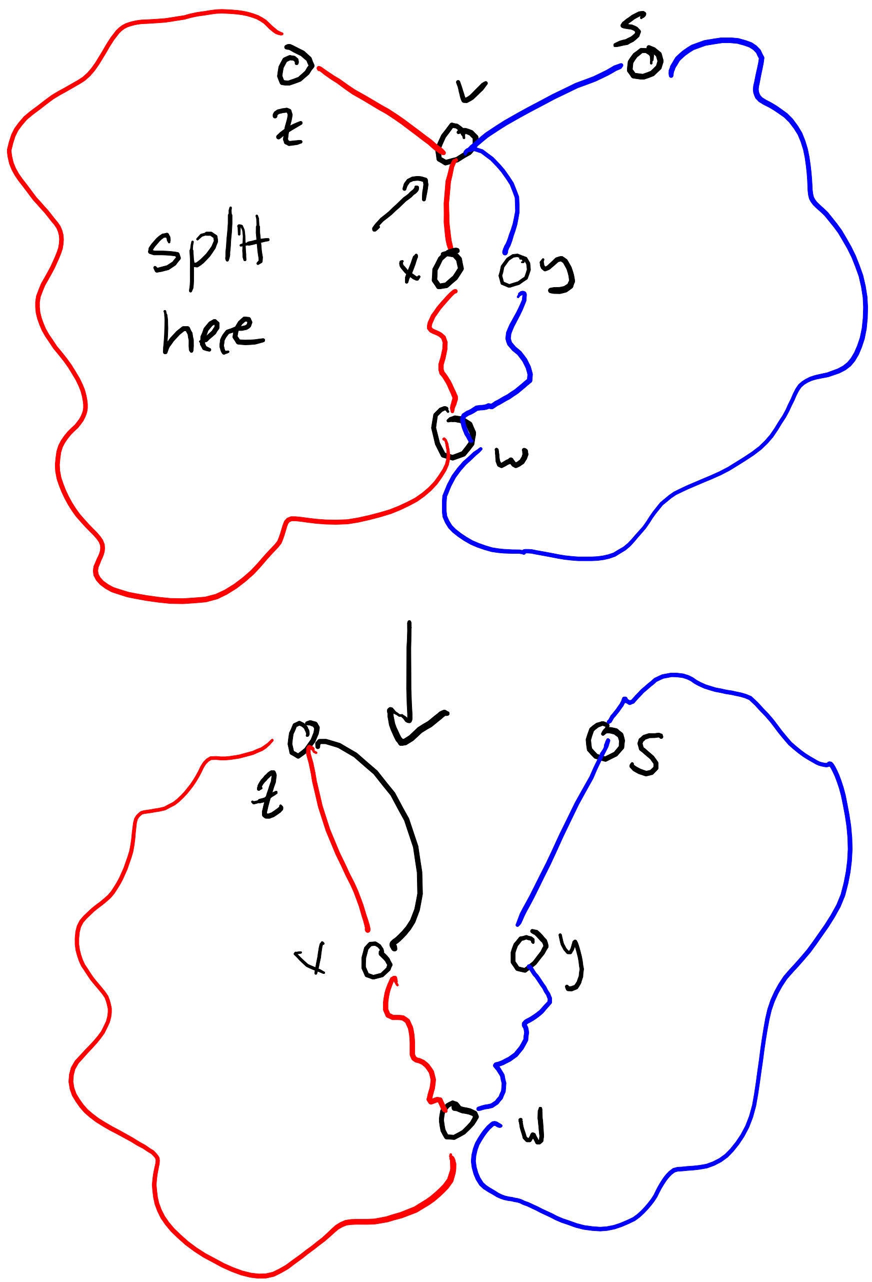


Take a cover of odd cycles maximizing the number of odd triangles.

Claim: If two odd circuits C, D meet twice, one is a triangle.

Proof: Start with previous claim. Now split in a way that preserves C & D and weak-6-connectivity.

Assume for a contradiction that neither C nor D are triangles.



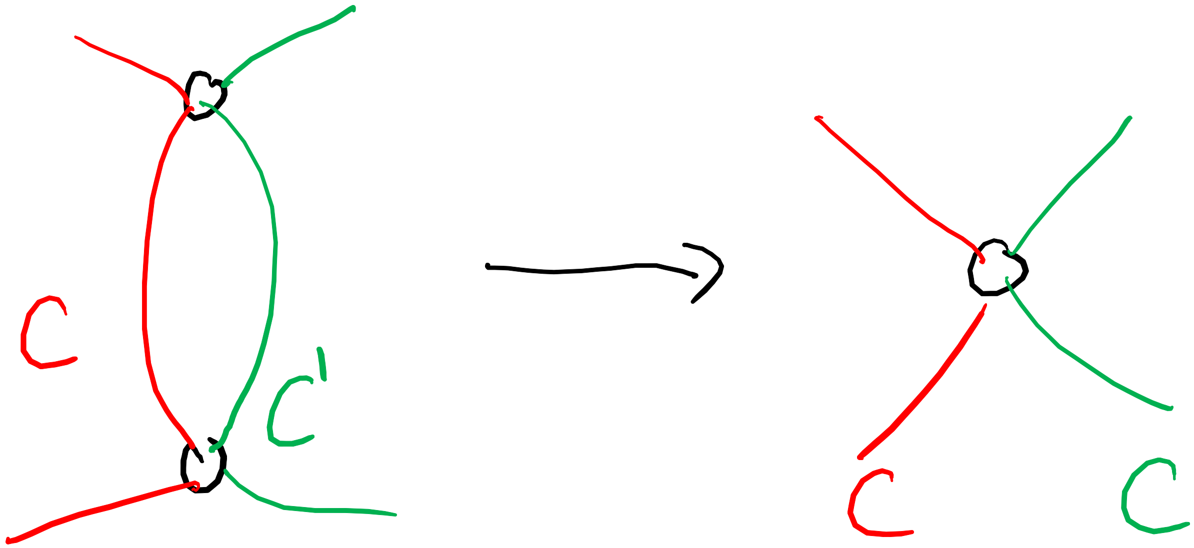
Now weak but not internal 6-connected graphs have parallel pairs.

So by minimality one of {z,x} or {s,y} is in a parallel pair.

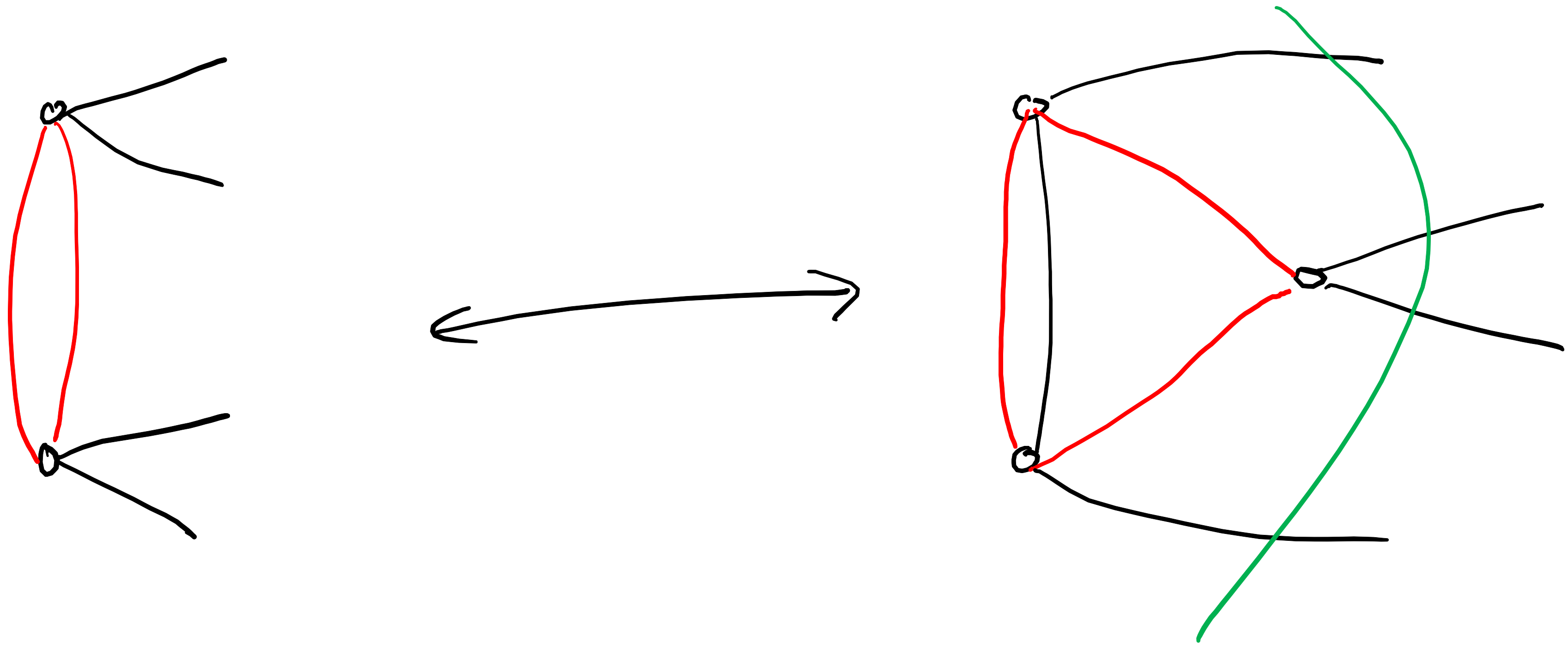
We may assume {z,x} is (it's symmetric)

Now as C is not a triangle, it's not the {z,x} parallel pair.

By covering claim, then there are two circuits C and C' running through the pair.



Split again, preserving parity. This gets rid of a parallel pair while not introducing any new ones – original graph weakly 6-connected.



not weak 6-connected

So the resulting graph is an internally-6-connected immersion with the same number of odd circuits

#

Weak Bouchet's Theorem: Excluded minors have ≤ 10 vertices.

Proof: Take a cycle cover with as many odd triangles as possible. Note $|V(R)| = |V(G)|-1$.

If 3 triangles, we're done, as every vertex sees 1 triangle.

If 2, and if $|V(G)| \geq 10$, $|V(R)| \geq 9$. Hence there are 3 vertices covered by the remaining two cycles. Clearly they meet twice so one's a triangle.

Similar arguments work for 1 / 0 triangles.

$\boxed{///}$